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RELATIONSHIP BETWEEN THE STIFFLY WEIGHTED PSEUDOINVERSE AND MULTI-LEVEL CONSTRAINED PSEUDOINVERSE [*1)]

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Abstract

It is known that for a given matrix A of rank r, and a set $\mathcal{D}$ of positive diagonal matrices, $\sup_{W\in\mathcal{D}} \|(W^{\frac{1}{2}}A)^{\dagger}W^{\frac{1}{2}}\|_2 = (\min_i \sigma_+(A^{(i)}))^{-1}$, in which $(A^{(i)})$ is a submatrix of A formed with $r = (\text{rank}(A))$ rows of A, such that $(A^{(i)})$ has full row rank r. In many practical applications this value is too large to be used.

In this paper we consider the case that both A and $W(\in \mathcal{D})$ are fixed with W severely stiff. We show that in this case the weighted pseudoinverse $(W^{\frac{1}{2}}A)^{\dagger}W^{\frac{1}{2}}$ is close to a multi-level constrained weighted pseudoinverse therefore $\|(W^{\frac{1}{2}}A)^{\dagger}W^{\frac{1}{2}}\|_2$ is uniformly bounded. We also prove that in this case the solution set the stiffly weighted least squares problem is close to that of corresponding multi-level constrained least squares problem.

Mathematics subject classification: 15A09, 15A12, 65F20.
Key words: Weighted least squares, Stiff, Multi-Level constrained pseudoinverse.

1. Introduction

In this paper we are concerned with the stiffly weighted least squares (stiffly WLS) problem

$$\min_x \|W^{\frac{1}{2}}(Ax-b)\|_2 = \min_x \|D(Ax-b)\|_2 \tag{1}$$

and related weighted pseudoinverse $A_W^{\dagger} \equiv (W^{\frac{1}{2}}A)^{\dagger}W^{\frac{1}{2}}$, where $\|\cdot\| \equiv \|\cdot\|_2$ denotes the Euclidean vector norm or subordinate matrix norm, $A \in \mathbf{C}^{m\times n}, b \in \mathbf{C}^m$ are known coefficient matrix and observation vector, respectively,

$$D = \text{diag}(d_1, d_2, \cdots, d_m) = \text{diag}(w_1^{\frac{1}{2}}, w_2^{\frac{1}{2}}, \cdots, w_m^{\frac{1}{2}}) = W^{\frac{1}{2}} \tag{2}$$

is the weight matrix. WLS problem Eq. (1) with extremely ill-conditioned weight matrix W (in this case Björck [3] called W stiff weight matrix), where the scalar factors $w_1, \cdots, w_m$ vary widely in size arise, e.g., in electronic network, certain classes of finite element problems, interior-point method for constrained optimization (e.g., see [8, 15]), and for solving the equality constrained least squares problem by the method of weighting [16, 1, 14].

In the case that W is severely stiff, it is not at all apparent that an accurate numerical solution to Eq. (1) is possible, since ill-conditioning in W presumably means extreme sensitivity to roundoff errors, because in standard numerical analysis, error bounds of the solutions to Eq. (1) have a weighted condition number $\kappa(W^{\frac{1}{2}}A) = \|W^{\frac{1}{2}}A\|\|(W^{\frac{1}{2}}A)^{\dagger}\|$ as a factor so that when W becomes ill-conditioned the condition number would become unbounded.

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On the other hand, one can define a new condition number $\kappa = \|A\|\|A_W^\dagger\|$. If $\|A_W^\dagger\|$ is uniformly bounded, then κ would be uniformly bounded.

Stewart [13] obtained an upper bound of scaled projections when $A \in R^{m\times n}$ has full column rank and weight matrices W range over a set $\mathcal{D}$ of positive diagonal matrices. Liu and Xu [10] then proved that this upper bound for scaled projection is indeed the supremum. Wei [19], Forsgren [6], Wei [20] respectively have obtained the supremum of weighted pseudoinverses when weight matrices W range over $\mathcal{D}$, or a set $\mathcal{P}$ of real symmetric diagonal dominant semi-positive matrices. Forsgren [6] and Wei [20] have also extended the results to constrained weighted pseudoinverses. For more detailed description on this topic, we refer to [21].

In practical applications, the supremum [19, 20]

$$\sup_{W\in\mathcal{D}} \|A_W^\dagger\| = \frac{1}{\min \sigma_+(A^{(i)})} \tag{3}$$

sometimes may be too large to be of practical usefulness. For instance, suppose

$$A = \begin{pmatrix} 1 & 0 \\ \delta & 0 \\ 0 & 1 \end{pmatrix},\ W_0 = \operatorname{diag}(w_1, w_1, w_3),$$

where $w_1 > w_3 > 0$ are arbitrary, and $0 < \delta \ll 1$. Then

$$\|A_{W_0}^\dagger\| = 1 \quad\text{and}\quad \sup_{W\in\mathcal{D}} \|A_W^\dagger\| = \frac{1}{\delta} \gg 1.$$

This example rises a question: if the weight matrix W is given and is very ill-conditioned, does exist an upper bound of $\|A_W^\dagger\|$ which is of moderate size?

In this paper we will study the above question. Without loss of generality, we make the following notation and assumptions for A and W.

Assumption 1.1. *The matrices A and W in Eq. (1) satisfy the following conditions: $\|A(i,:)\| \equiv \|(a_{i1}, a_{i2}, \cdots, a_{in})\|$ have the same order for $i = 1, \cdots, m$, $w_1 > w_2 > \cdots > w_k > 0$, $m_1 + m_2 + \cdots + m_k = m$, and we denote*

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_k \end{pmatrix} \begin{matrix} m_1 \\ \vdots \\ m_k \end{matrix},\ C_j = \begin{pmatrix} A_1 \\ \vdots \\ A_j \end{pmatrix}, \quad j = 1, \cdots, k, \tag{4}$$

$$\begin{aligned} &W = \operatorname{diag}(w_1 I_{m_1}, w_2 I_{m_2}, \cdots, w_k I_{m_k}), \\ &0 < \epsilon_{ij} \equiv \tfrac{w_i}{w_j} \ll 1,\ \textit{for } 1 \le j < i \le k \textit{ so } \epsilon = \max_{1\le j<k}\{\epsilon_{j+1,j}\} \ll 1. \end{aligned} \tag{5}$$

We also set

$$P_0 = I_n,\ P_j = I - C_j^\dagger C_j,\ \operatorname{rank}(C_j) = r_j,\ j = 1, \cdots, k. \tag{6}$$

Vavasis and Ye [17] studied interior-point method for solving linear programming problem, in which the matrices A and W basically satisfy Assumption 1.1.

The paper is organized as follows. In §2 we will derive several equivalent formulas of the stiffly weighted pseudoinverse; in §3 we will derive the multi-level constrained pseudoinverse and corresponding multi-level constrained least squares (MCLS) problem; in §4 we will prove that the stiffly weighted pseudoinverse is indeed close to the multi-level constrained pseudoinverse therefore is uniformly bounded; in §5 we will deduce upper bounds of difference of the solutions between of the stiffly WLS problem and the MCLS problem; finally in §6 we will conclude the paper with some remarks.