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BOUNDEDNESS AND ASYMPTOTIC STABILITY OF MULTISTEP METHODS FOR GENERALIZED PANTOGRAPH EQUATIONS *1)

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Abstract

In this paper, we deal with the boundedness and the asymptotic stability of linear and one-leg multistep methods for generalized pantograph equations of neutral type, which arise from some fields of engineering. Some criteria of the boundedness and the asymptotic stability for the methods are obtained.

Mathematics subject classification: 65L06, 65L20.
Key words: Boundedness, Asymptotic stability, Multistep methods, Generalized pantograph equations.

1. Introduction

Consider generalized pantograph equations

$$\begin{cases} Y'(t) = AY(t) + BY(pt) + CY'(pt), \quad t > 0, \\ Y(0) = Y_0, \end{cases} \tag{1.1}$$

where $A, B, C \in \mathbb{C}^{d\times d}, p \in (0,1)$. The above equations possess numerous applications in some fields of engineering (cf. [1]), and therefore has induced much research (cf. [1]-[9]). In particular, Iserles [1, 2] and Liu [3] proved respectively that

[I] (1.1) *has a unique solution* $Y(t)$ *on space* $\mathbb{C}^{N+1}[0,\infty)$, *provided that* $p^N\|C\| < 1$ *for any given norm* $\|\bullet\|$ *and matrices* $I - p^nC \ \ (n = 0, 1, \dots, N-1)$ *are nonsingular;*

[IL] *the solution* $Y(t)$ *of* (1.1) *is asymptotically stable (i.e.,* $\lim_{t\to+\infty} Y(t) = 0$*), provided*

$$\rho(A^{-1}B) < 1 \quad \text{and} \quad \alpha(A) < 0, \tag{1.2}$$

where $\rho(\bullet)$ *denotes the spectral radius and* $\alpha(\bullet)$ *the spectral abscissa (i.e., the maximal real part of the eigenvalues of the corresponding matrix).*

A remarkable fact is that there exist some differences between equations (1.1) and delay equations of the form

$$\begin{cases} Y'(t) = AY(t) + BY(t-\tau) + CY'(t-\tau), \quad t > 0, \\ Y(t) = Y_0(t), \quad -\tau \le t \le 0. \end{cases} \tag{1.3}$$

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These differences are embodied mainly in the smoothness of solutions and the numerical treatment of equations (cf. [1]-[9]). The most significant difference is in storage (cf.[5, 6]). Namely, when solving (1.1) with a numerical method, we first need to resolve the storage problem for the existence of infinite delays in (1.1), while the computation for (1.3) will not suffer such a difficulty, because there is only a constant-delay in it. To overcome the computational storage problem for (1.1), Liu [4] (see also Koto [9]) considered a transformation of the form

$$y(t) = Y(\exp(t)), \quad t \geq t_0 + \ln p \quad (t_0 > 0), \tag{1.4}$$

which converts (1.1) into the equations

$$\begin{cases} y'(t) = \exp(t)Ay(t) + \exp(t)By(t + \ln p) + p^{-1}Cy'(t + \ln p), & t > t_0, \\ y(t) = Y(\exp(t)), \quad t_0 + \ln p \leq t \leq t_0, \end{cases} \tag{1.5}$$

where $Y(t) \ (0 < t \leq \exp(t_0))$ can be obtained numerically by the assigned numerical methods to (1.1).

Making use of the above technique, Liu [4] and Koto [9] studied the stability of θ-methods and Runge-Kutta methods for (1.1), respectively. We note that the previous research dealt mainly with one-step methods while multistep methods have not been involved. Hence, in the present paper, we focus on the boundedness and the asymptotic stability of linear and one-leg multistep methods. The corresponding results can be found in section 3 and section 4. In section 5, some examples are given to illustrate the applicability of the obtained theoretical results.

2. Multistep Methods

For the initial value problems of ordinary differential equations

$$\begin{cases} x'(t) = f(t, x(t)), \quad t \geq 0, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

two standard discretization schemes are the linear multistep methods

$$\sum_{j=0}^{k} \alpha_j x_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{2.2}$$

and the corresponding one-leg methods

$$\sum_{j=0}^{k} \alpha_j x_{n+j} = hf(\sum_{j=0}^{k} \beta_j t_{n+j}, \sum_{j=0}^{k} \beta_j x_{n+j}). \tag{2.3}$$

They can be characterized by the polynomials

$$P(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j, \quad Q(\xi) = \sum_{j=0}^{k} \beta_j \xi^j, \quad \xi \in \mathbb{C},$$

where $\alpha_j, \ \beta_j \ (j = 0, 1, \ldots, k)$ are real constants with

$$P(1) = 0, \quad P'(1) = Q(1) = 1. \tag{2.4}$$

Motivated by an idea of Hu and Mitsui [11] (see also [13]), we adapt (2.2) and (2.3) to (1.5), respectively, and thus obtain two computational schemes:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j \exp(t_{n+j})(Ay_{n+j} + By_{n+j-m}) + p^{-1}C \sum_{j=0}^{k} \alpha_j y_{n+j-m}, \tag{2.5}$$