Journal of Computational Mathematics, Vol.22, No.4, 2004, 581–592.

FROM ENERGY IMPROVEMENT TO ACCURACY ENHANCEMENT: IMPROVEMENT OF PLATE BENDING ELEMENTS BY THE COMBINED HYBRID METHOD *1)

Xiao-ping Xie
(*Mathematical College, Sichuan University, Chengdu 610064, China*)

Abstract

By following the geometric point of view in mechanics, a novel expression of the combined hybrid method for plate bending problems is introduced to clarify its intrinsic mechanism of enhancing coarse-mesh accuracy of conforming or nonconforming plate elements. By adjusting the combination parameter $\alpha \in (0,1)$ and adopting appropriate bending moments modes, reduction of energy error for the discretized displacement model leads to enhanced numerical accuracy. As an application, improvement of Adini's rectangle is discussed. Numerical experiments show that the combined hybrid counterpart of Adini's element is capable of attaining high accuracy at coarse meshes.

Mathematics subject classification: 65N12, 65N30.
Key words: Finite element, Combined hybrid, Energy error.

1. Introduction

The combined hybrid finite element method [6,7,8,9] is capable of remarkably enhancing coarse-mesh accuracy of conventional lower order elements for linear elasticity problems. The 4-node plane quadrilateral CH(0-1) proposed in [9] is a successful example.

By following the geometric point of view in mechanics, a novel expression of the combined hybrid method was introduced in [10] to clarify its intrinsic mechanism of enhancing coarse-mesh accuracy and stability of lower order displacement schemes for linear elasticity problems. For a fixed coarse mesh and a given stress mode, e.g. the piecewise constant stress mode, one can adjust the energy of the finite element model such that the energy error reduces to zero by optimizing the combined parameter α and by adding energy compatible bubble displacements to the given conforming displacements. It was shown by numerical experiments that the smaller the energy error is, the higher numerical accuracy will be, and that combined hybrid schemes without energy error are of high accuracy at coarse meshes. This accuracy criterion of schemes at coarse meshes is different from the gradual convergence of the $h-$version and the $p-$version, i.e. it does not require the mesh size h being smaller or the degree p of elements being bigger for the combined hybrid method to achieve higher accuracy.

In the reference [11], the combined hybrid finite element method was applied to 4th-order plate bending problems. It was shown that the resultant schemes are stabilized, i.e., the convergence of the schemes is independent of inf-sup conditions and any other patch test. Then the deflection interpolant and the bending moments approximation can be chosen independently, which provides possibility of optimizing bending moments modes so as to obtain accurate plate elements.

Based on [11], the present paper is devoted to a further analysis of the mechanism of enhancing coarse-mesh accuracy of conventional plate elements of the combined hybrid method.

* Received May 26, 2002.
1) This work was supported by the National Natural Science Fund (10171073) and the Nationmal Tianyuan Youth Fund (TY10126027).

By adopting rational bending moments modes and adjusting the combination parameter $\alpha \in (0,1)$, energy error of the discretized scheme can be reduced, and then an enhanced numerical accuracy at coarse meshes can be acquired. As an application, improvement of Adini's rectangle is discussed and numerical experiments show that the combined hybrid counterpart of Adini's element is capable of attaining high coarse-mesh accuracy.

In what follows the letter C will represent a constant which is independent of the mesh size $h = \max_K \{h_K\}$ and may be different at its each occurrence.

2. Combined Hybrid Variational Principle

Considering the following plate bending problem:

$$\begin{cases} \mathbf{divdiv}\sigma = f, & in\ \Omega, \\ \sigma = \mathbf{m}(\mathbf{D}_2 u), & in\ \Omega, \\ u = \nabla u \cdot \mathbf{n} = 0, & on\ \Gamma = \partial\Omega. \end{cases} \tag{2.1}$$

where $\Omega \subset \Re^2$ is a bounded open set, u represents vertical deflection, σ the bending moments, and $\mathbf{n}$ the outer normal unit vector along Γ. The operators $\mathbf{divdiv}, \mathbf{D}_2$ and $\mathbf{m}$ are defined respectively as follows:

$$\mathbf{divdiv}\tau = \partial_{11}\tau_{11} + 2\partial_{12}\tau_{12} + \partial_{22}\tau_{22},$$

$$\mathbf{D}_2 v = \begin{pmatrix} \partial_{11}v & \partial_{12}v \\ \partial_{12}v & \partial_{22}v \end{pmatrix},$$

$$\mathbf{m}(\tau) = \begin{pmatrix} \tau_{11} + \nu\tau_{22} & (1-\nu)\tau_{12} \\ (1-\nu)\tau_{12} & \nu\tau_{11} + \tau_{22} \end{pmatrix}$$

for any symmetric tensor $\tau = (\tau_{ij}), i, j = 1, 2$, and $\nu \in (0, 0.5)$ denotes the Poisson's coefficient, $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j},\ i, j = 1, 2.$

As shown in the reference [11], the combined hybrid variational principle equivalent to the problem (2.1) reads as:

$$\inf_{(v,v_c)\in U\times U_c} \sup_{\tau\in\mathbf{V}} \{\tfrac{1-\alpha}{2} d(v,v) - f(v) - b_1(\tau, v - v_c) + \alpha[b_2(\tau, v) - \tfrac{1}{2} a(\tau,\tau)]\} \tag{2.2}$$

where

$$U := \{v \in \prod_{K\in T_h} H^2(K); u = \nabla u \cdot \mathbf{n} = 0, on\ \Gamma\},$$

$$\mathbf{V} := \prod_{K\in T_h} H(\mathbf{divdiv}; K) = \prod_{K\in T_h} \{\tau \in (L^2(K))^4_s; \mathbf{divdiv}\tau \in L^2(K)\}$$

and

$$U_c := H^2_0(\Omega) / \prod_{K\in T_h} H^2_0(K)$$

are respectively the deflection space, the symmetric bending moments vector space and the interelemental boundary deflection space, $T_h = \{K\}$ denotes a regular subdivision of Ω, with mesh diameter h_K for any $K \in T_h$, $(L^2(K))^4_s$ the space of square integrable 2×2 symmetric tensors, and