Journal of Computational Mathematics, Vol.22, No.4, 2004, 611–621.

A MULTI-SYMPLECTIC SCHEME FOR RLW EQUATION *)

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Abstract

The Hamiltonian and multi-symplectic formulations for RLW equation are considered in this paper. A new twelve-point difference scheme which is equivalent to multi-symplectic Preissmann integrator is derived based on the multi-symplectic formulation of RLW equation. And the numerical experiments on solitary waves are also given. Comparing the numerical results for RLW equation with those for KdV equation, the inelastic behavior of RLW equation is shown.

Mathematics subject classification: 65N15.
Key words: Multi-Symplectic Scheme, RLW Equation.

1. Introduction

We consider the regularized long-wave (RLW) equation [1]

$$u_t + (V'(u))_x - \sigma u_{xxt} = 0, \tag{1}$$

where $V(u) = \frac{1}{6}u^3$. The equation was first put forward by Peregrine (1966) [2] to describe the development of long wave behaviour. B. Benjamin, J. L. Bona and J. J. Mahoney (1972) [1] studied in detail the existence, uniqueness and stability of solutions of equation (1) and considered it as a more suitable posed model for long wave. (1) was called as BBM equation by P. J. Olver (1979) [6], who studied the conservation laws of equation (1) and proved that it possesses only three independent conservation laws. At the same year, the comparative numerical experiments between RLW equation and KdV equation were given by M. E. Alexander and J. L. Morris [5], the results showed that the interaction of two-solitary wave is inelastic during evolution process. It was known very well that KdV equation as a completely integrable model has infinite family of conservation laws, so the interaction between solitons is elastic. Therefore, it was clear that RLW equation and KdV equation describe different physical phenomena.

The purpose of this paper is to present a multi-symplectic formulation for RLW equation, then by using multi-symplectic Preissmann scheme to derive a new twelve-point scheme. At last we give the comparative numerical experiments on solitary waves for RLW equation and KdV equation by using the new twelve-point scheme given in this paper and the twelve-point scheme given by Pingfu Zhao and Mengzhao Qin [11]. These numerical results obtained in this paper further confirm the inelastic behaviour of the RLW equation and show that the multi-symplectic twelve-point scheme given in [11] treats the KdV equation more successfully than the Galerkin method got in [5]. In addition, we simulate the interaction of three solitary waves.

An outline of this paper is as follows. In section 2, we present the Hamiltonian formulation and the multi-symplectic formulation for RLW equation. In section 3, we derive a new twelve -point difference scheme based on multi-symplectic Preissmann scheme. Numerical experiments are presented in section 4.

* Received January 21, 2002; final revised August 27, 2003.

2. Symplectic and Multi-symplectic Formulations of RLW Equation

In this section, we discuss symplectic and multi-symplectic formulations for RLW equation. First, (1) can be written as the classical Hamiltonian formulation

$$u_t = -(1-\sigma D^2)^{-1} D \frac{\delta H}{\delta u},$$

where the Hamilton operator is $-(1-\sigma D^2)^{-1}D,\ D = \frac{\partial}{\partial x}$, and the Hamiltonian function is $H = \int \frac{1}{6} u^3 dx$.

Let $u = \varphi_x$, (1) can be rewritten as the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\delta L}{\delta \varphi_t} = \frac{\delta L}{\delta \varphi}$$

with Lagrangian functional

$$L(\varphi, \varphi_t) = \int (\frac{1}{2}\varphi_t\varphi_x + \frac{1}{2}\sigma\varphi_{xt}\varphi_{xx} + \frac{1}{6}\varphi_x^3)dx.$$

Here, the canonical 1-form on infinite dimensional functional space TQ with coordinates (φ, φ_t) can be considered as

$$\theta = \frac{\delta L}{\delta \varphi_t} d\varphi = \frac{1}{2}(1-\sigma D^2)\varphi_x d\varphi = \frac{1}{2}(1-\sigma D^2) u D^{-1} du.$$

Now, the canonical two-form on TQ is

$$\Omega = -d\theta = -\frac{1}{2}(1-\sigma D^2) du \wedge D^{-1} du = \frac{1}{2}(\sigma du \wedge du_x - du \wedge D^{-1} du).$$

Second, we can describe (1) by using multi-symplectic language. According to the multi-symplectic concept introduced by Bridges [6], the RLW equation can be reformulated as the following

$$Mz_t + Kz_x = \bigtriangledown_z S(z), \quad z \in R^5, \quad (x,t) \in R^2, \tag{2}$$

with state variable $z = (\varphi, u, v, w, p)^T$ and the Hamiltonian function

$$H = up - \frac{1}{6}u^3 + \frac{1}{2}\sigma v w,$$

where

$$M = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2}\sigma & 0 & 0 \\ 0 & \frac{1}{2}\sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad K = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\frac{1}{2}\sigma & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sigma & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$u = \varphi_x,\ v = u_x,\ w = u_t,\ p = \frac{1}{2}\varphi_t + \frac{1}{2}\varphi_x^2 - \sigma\varphi_{xxt}. \tag{3}$$

Taking exterior derivative on the two side of (2), we have

$$Mdz_t + Kdz_x = \bigtriangledown_z^2 S(z) dz.$$

Then take the wedge product with dz on the two side of the equality above, a multi-symplectic conservation law can be derived

$$\frac{\partial}{\partial t}(du \wedge d\varphi + \sigma dv \wedge du) + \frac{\partial}{\partial x}(2dp \wedge d\varphi + \sigma dw \wedge du) = 0. \tag{4}$$