Journal of Computational Mathematics, Vol.22, No.5, 2004, 633–640.

A STABILITY THEOREM FOR CONSTRAINED OPTIMAL CONTROL PROBLEMS *

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Abstract

This paper presents the stability of difference approximations of an optimal control problem for a quasilinear parabolic equation with controls in the coefficients, boundary conditions and additional restrictions. The optimal control problem has been convered to one of the optimization problem using a penalty function technique. The difference approximations problem for the considered problem is obtained. The estimations of stability of the solution of difference approximations problem are proved. The stability estimation of the solution of difference approximations problem by the controls is obtained.

Mathematics subject classification: 49M29, 49M30 , 49J20, 49K20.
Key words: Optimal control, Quasilinear Parabolic equations, Penalty function methods, Finite difference method, Stability estimations.

1. Introduction

Optimal control problems for partial differential equations are currently of much interest. A large amount of the theoretical concept which governed by quasilinear parabolic equations [1-4] has been investigated in the field of optimal control problems. These problems have dealt with the processes of hydro- and gasdyn amics, heatphysics, filtration, the physics of plasma and others [5-6]. Difference methods of solution of optimal control problems for partial differential equations are investigated comparatively small [7-11]. In this paper, the stability of difference approximations of an optimal control problem for a quasilinear parabolic equation with controls in the coefficients, boundary conditions and additional restrictions. The optimal control problem has been convered to one of the optimization problem using a penalty function technique. The difference approximations problem for the considered problem is obtained. The estimations of stability of the solution of difference approximations problem are proved. The stability estimation of the solution of difference approximations problem by the controls is obtained.

Let $\Omega = \{(x,t) : x \in D = (0,l), t \in (0,T)\}$ where l, T are given positive numbers. Now, we need to introduce some functional spaces as follows:
1) $L_2(D)$ is a Banach space which consisting of all measurable functions on D with the norm $\|z\|_{L_2(D)} = [\int_D \ |z|^2 \ dx \ ]^{\frac{1}{2}}$. 2) $L_2(0,l)$ is a Hilbert space which consisting of all measurable functions on $(0,l)$ with

$$\langle z_1, z_2\rangle_{L_2(0,l)} = \int_0^l z_1(x)z_2(x)dx, \|z\|_{L_2(0,l)} = \sqrt{\langle z, z\rangle_{L_2(0,l)}}.$$

3) $L_2(\Omega)$ is a Hilbert space which consisting of all measurable functions on Ω with

$$\langle z_1, z_2\rangle_{L_2(\Omega)} = \int_0^l \int_0^T z_1(x,t)z_2(x,t)dxdt, \|z\|_{L_2(\Omega)} = \sqrt{\langle z, z\rangle_{L_2(\Omega)}}.$$

* Received May 12, 2002; final revised Febreary 3, 2004.

4) $W_2^{1,0}(\Omega)=\{z\in L_2(\Omega)$ *and* $\frac{\partial z}{\partial x}\in L_2(\Omega)\}$ is a Hilbert space with

$$\|z\|_{W_2^{1,0}(\Omega)}=\int_\Omega\ [\ z_1z_2+\frac{\partial z_1}{\partial x}\frac{\partial z_2}{\partial x}]\ dx\ \ dt$$

$$\langle z_1,z_2\rangle_{W_2^{1,0}(\Omega)}=[\|z\|^2_{L_2(\Omega)}+\|\frac{\partial z}{\partial x}\|^2_{L_2(\Omega)}]^{\frac{1}{2}}.$$

5) $W_2^{1,1}(\Omega)=\{z\in L_2(\Omega)$ *and* $\frac{\partial z}{\partial x}\in L_2(\Omega),\frac{\partial z}{\partial t}\in L_2(\Omega)\}$ is a Hilbert space with

$$\|z\|_{W_2^{1,1}(\Omega)}=\int_\Omega\ [\ z_1z_2+\frac{\partial z_1}{\partial x}\frac{\partial z_2}{\partial x}+\frac{\partial z_1}{\partial t}\frac{\partial z_2}{\partial t}]\ dx\ \ dt$$

$$\langle z_1,z_2\rangle_{W_2^{1,1}(\Omega)}=[\|z_1\|^2_{L_2(\Omega)}+\|\frac{\partial z}{\partial x}\|^2_{L_2(\Omega)}+\|\frac{\partial z}{\partial t}\|^2_{L_2(\Omega)}]^{\frac{1}{2}}.$$

6) $V_2(\Omega)$ is a Banach space consisting of elements the space $W_2^{1,0}(\Omega)$ with the norm

$$\|z\|_{V_2(\Omega)}=vraimax_{0\le t\le T}\|z(x,t)\|_{L_2(D)}+(\int_\Omega|\frac{\partial z}{\partial x}|^2)^{\frac{1}{2}}.$$

7) $V_2^{1,0}(\Omega)$ is a subspace of $V_2(\Omega)$, the elements of which have in sections $D_t=\{(x,\tau):x\in D,\tau=t\}$ traces from $L_2(D)$ at all $t\in[0,T]$,continuously changing from $t\in[0,T]$ in the norm $L_2(D)$.

2. Problem Formulation

Let $V=\{v:v=(v_1,v_2,...,v_N)\in E_N,\|v\|_{E_N}\le R\}$, where $R>0$ are given numbers. We consider the heat exchange process described by the equation

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}(\lambda(u,v)\frac{\partial u}{\partial x})+B(u,v)\frac{\partial u}{\partial x}=f(x,t),(x,t)\in\Omega \tag{1}$$

with initial and boundary conditions

$$u(x,0)=\phi(x),x\in D \tag{2}$$

$$\lambda(u,v)\frac{\partial u}{\partial x}|_{x=0}=Y_0(t),\lambda(u,v)\frac{\partial u}{\partial x}|_{x=l}=Y_1(t),0\le t\le T \tag{3}$$

where $\phi(x)\in L_2(D)$, $Y_0(t),Y_1(t)\in L_2(0,T)$ and $f(x,t)$ is given function. Besides, the functions $\lambda(u,v),B(u,v)$ are continuous on $(u,v)\in[r_1,r_2]$ x E_N, have continuous derivatives in u and $\forall(u,v)\in[r_1,r_2]$ x E_N, the derivative $\frac{\partial\lambda(u,v)}{\partial u}$, $\frac{\partial B(u,v)}{\partial u}$ are bounded. Here r_1,r_2 are given numbers.

On the set V, under the conditions (1)-(3) and additional restrictions

$$\nu_0\le\lambda(u,v)\le\mu_0,\qquad\nu_0\le B(u,v)\le\mu_0,\qquad r_1\le u(x,t)\le r_2 \tag{4}$$

is required to minimize the function

$$J_\alpha(v)=\beta_0\int_0^T[u(0,t)-f_0(t)]^2dt+\beta_1\int_0^T[u(l,t)-f_1(t)]^2dt+\alpha\|v-\omega\|^2_{E_N} \tag{5}$$

where $f_0(t)$, $f_1(t)\in L_2(0,T)$ are given functions, $\alpha\ge0$, $\nu_0,\mu_0>0$, $\beta_0\ge0$, $\beta_1\ge0$, $\beta_0+\beta_1\ne0$ are given numbers, $\omega\in E_N$ is also given: $\omega=(\omega_1,\omega_2,...,\omega_N)$.

Definition 1. *The problem of finding a function $u=u(x,t)\in V_2^{1,0}(\Omega)$ from conditions (1)-(4) at given $v\in V$ is called the reduced problem.*

Definition 2. *The solution of the reduced problem (1)-(4) corresponding to the $v\in V$ is a function $u(x,t)\in V_2^{1,0}(\Omega)$ and satisfies the integral identity*

$$\int_0^l\int_0^T[u\frac{\partial\eta}{\partial t}-\lambda(u,v)\frac{\partial u}{\partial x}\frac{\partial\eta}{\partial x}-B(u,v)\frac{\partial u}{\partial x}\eta+\eta f(x,t)]dxdt=$$

$$-\int_0^l\phi(x)\eta(x,0)dx-\int_0^T\eta(0,t)Y_0(t)dt+\int_0^T\eta(l,t)Y_1(t)dt, \tag{6}$$

$\forall\quad\eta=\eta(x,t)\in W_2^{1,1}(\Omega)$ *and* $\eta(x,T)=0$.