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A LOCKING-FREE SCHEME OF NONCONFORMING RECTANGULAR FINITE ELEMENT FOR THE PLANAR ELASTICITY [*1)]

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Abstract

In this paper, the authors present a locking-free scheme of the lowest order nonconforming rectangle finite element method for the planar elasticity with the pure displacement boundary condition. Optimal order error estimate, uniformly for the Lamé constant $\lambda \in (0, \infty)$ is obtained.

Mathematics subject classification: 65N30, 73V05.
Key words: Locking-free, Planar elasticity, Nonconforming finite element method.

1. Introduction

For numerical solutions of the equations of linear isotropic planar elasticity , the conforming finite element method suffers a deterioration in performance as the Lamé constant $\lambda \to \infty$, i.e., as the material becomes incompressible. It is known as the phenomenon of locking (see [1], [2], [3] and [5]). By virtue of numerical analysis, the coefficient C_λ appearing in the error estimate of the conforming finite element approximation to the planar elasticity depends on the Lamé constant λ; and $C_\lambda \to \infty$ as $\lambda \to \infty$. Thus in order to overcome the phenomenon of locking, we need to construct a finite element method such that the numerical solutions of the finite element scheme converge to the true solution of the planar elasticity, as the mesh $h \to 0$, uniformly with respect to $\lambda \in (0, \infty)$.

There are some works on locking-free finite element methods for the planar elasticity. The Crouziex-Raviart element approximations to the pure displacement boundary value problem was considered in [2] and [3] by virtue of the standard finite element analysis. The pure traction boundary value problem was considered in [5] and [10] with triangular element approximations, [11] with quadrilateral element approximations and [12] with the NRQ_1 element approximations following the argument of [11] by the mixed finite element analysis. In the previous paper [9], we have considered a higher order nonconforming rectangular finite element method for the pure displacement boundary value problem of the planar elasticity. In the present paper, we derive and analyze the locking-free scheme of the lowest order rectangular finite element for the same problem. The locking-free finite element method for the pure traction boundary value problem of the planar elasticity will be considered in our forthcoming papers.

In the following section, we present the preliminary consideration of the locking-free finite element method for the planar elasticity. Then in section 3, we derive the lowest order locking-free scheme of nonconforming rectangular finite element. In section 4, optimal order of error estimate is obtained. At last, we end this paper with some numerical experiments.

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2. Preliminary

In this section, we present the general consideration of the locking-free finite element method for the planar elasticity with the pure displacement boundary condition.

Let $\Omega \subset R^2$ be a convex domain with the boundary $\partial\Omega$,

$$\begin{cases} -\mu\Delta\vec{u} - (\mu+\lambda)grad(div\vec{u}) = \vec{f} & in \quad \Omega \\ \vec{u} = 0 & on \quad \partial\Omega. \end{cases} \tag{2.1}$$

The corresponding variational problem is as follows

$$\begin{cases} \text{to find} \quad \vec{u} \in V, \text{ such that} \\ a(\vec{u},\vec{v}) = (\vec{f},\vec{v}) \quad \forall \vec{v} \in V, \end{cases} \tag{2.2}$$

where $V = (H_0^1(\Omega))^2$,

$$\begin{aligned} a(\vec{u},\vec{v}) &= \int_\Omega \{\mu\, grad\vec{u} : grad\vec{v} + (\mu+\lambda)(div\vec{u})\,(div\vec{v})\}dx \\ &\doteq \mu \int_\Omega \{gradu_1 \cdot gradv_1 + gradu_2 \cdot gradv_2\}dx \\ &\quad +(\mu+\lambda)\int_\Omega (div\vec{u})(div\vec{v})dx, \end{aligned} \tag{2.3}$$

$$(\vec{f},\vec{v}) = \int_\Omega \vec{f}\cdot\vec{v}dx, \tag{2.4}$$

and $\lambda \in (0,\infty), \mu \in [\mu_1,\mu_2], 0 < \mu_1 < \mu_2$, are the Lamé constants. Since the bilinear form $a(\cdot,\cdot)$ (2.3) is V-elliptic, there exists an unique solution of the problem (2.2). Now we consider the conforming finite element approximation to the problem (2.2). For the sake of simplicity, we assume that Ω is a convex polygon. Let $\Im_h$ be the regular triangulation of $\Omega, V_h \subset V$ be the conforming finite element space with respect to $\Im_h$, then the finite element approximation to the problem (2.2) is as follows:

$$\begin{cases} \text{to find} \quad \vec{u}_h \in V_h, \text{ such that} \\ a(\vec{u}_h,\vec{v}_h) = (\vec{f},\vec{v}_h) \quad \forall \vec{v}_h \in V_h. \end{cases} \tag{2.5}$$

The following error estimate holds

Theorem 2.1 (see[9]). *Assume that $\vec{u} \in (H^2(\Omega))^2$ and $\vec{u}_h$ are the solutions of the problems (2.2) and (2.5) respectively, then*

$$\|\vec{u}-\vec{u}_h\|_{1,\Omega} \le C_\lambda \cdot h|\vec{u}|_{2,\Omega}, \quad C_\lambda = C\sqrt{2\mu+\lambda}, \tag{2.6}$$

where $C = Const. > 0$ is independent of h and λ.

From theorem 2.1, it can be seen that the solution $\vec{u}_h$ of the conforming finite element approximation (2.5) converges to the solution $\vec{u}$ of the problem (2.2) as $h \to 0$, for each fixed λ; but we can not say any thing about the convergence of $\vec{u}_h$ when $\lambda \to \infty$. In fact, Brenner et al.[2] proved that the solution $\vec{u}_h$ of the conforming linear finite element approximation, with respect to triangulation $\Im_h$, might not converge to the solution $\vec{u}$ of the problem (2.2) when $\lambda \to \infty$. It is known as the phenomenon of locking. By the argument of [9], it can be seen that to overcome the locking, the crucial point is to construct a finite element space V_h, and an interpolation operator $\Pi_h : (H^1(\Omega))^2 \to V_h$, such that the following commutativity property holds

$$div\Pi_h\vec{u} = \gamma_h div\vec{u}, \tag{2.7}$$

where $\gamma_h : L^2(\Omega) \to W_h$ is another operator, and W_h is a piecewise polynomial space with lower degree than those in V_h; and the following error estimate is required:

$$\|div\vec{u} - \gamma_h(div\vec{u})\|_{0,\Omega} \le Ch|div\vec{u}|_{1,\Omega}. \tag{2.8}$$