Journal of Computational Mathematics, Vol.22, No.5, 2004, 681–698.

# ON THE CONVERGENCE OF WAVEFORM RELAXATION METHODS FOR LINEAR INITIAL VALUE PROBLEMS *1)

Jian-yu Pan Zhong-zhi Bai
(*LSEC, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

**Abstract**

We study a class of blockwise waveform relaxation methods, and investigate its convergence properties in both asymptotic and monotone senses. In addition, the monotone convergence rates between different pointwise/blockwise waveform relaxation methods resulted from different matrix splittings, and those between the pointwise and blockwise waveform relaxation methods are discussed in depth.

*Mathematics subject classification:* 65L05, 65F10.
*Key words*: Blockwise waveform relaxation method, Asymptotic and monotone convergence, Comparison results.

## 1. Introduction

The waveform relaxation method is a basic and efficient iteration technique for solving ordinary differential equations and differential-algebraic equations. It differs from classical iterative techniques in that it is a continuous-time method, iterating with functions in a functional space, and therefore is quite suitable for parallel computation. This kind of waveform relaxation method was first proposed by Lelarasmee, Ruehli and Sangiovanni-Vincentelli[19] in VLSI-simulation, and was further studied and improved by many authors on both method models and convergence properties. For example, Nevanlinna[23, 24] discussed the waveform relaxation method on finite interval in terms of Picard-Lindelöf iteration, Janssen and Vandewalle[18] studied the convolution SOR waveform relaxation methods, and Miekkala[20] studied the applications of the waveform relaxation method to differential-algebraic equations. In addition, Zubik-Kowal and Vandewalle[30] recently extended waveform relaxation technique to functional-differential equations. For further details we refer to [20, 13, 14, 17, 21, 22] and references therein.

However, so far as we know, most of these theoretical convergence results are about the pointwise waveform relaxation method, and there is few about its blockwise alternative.

In this paper, we will consider convergence properties of the blockwise waveform relaxation method for the linear initial value problems on the infinite interval $[0, +\infty)$ in both asymptotic and monotone senses. By making use of the block partition and the accelerated overrelaxation techniques[16], we first set up a kind of blockwise waveform accelerated overrelaxation method. This new method involves three arbitrary parameters, and therefore its convergence properties can be considerably improved by suitable adjustments of these parameters. In addition, a series of applicable and efficient blockwise waveform relaxation methods can be produced by various choices of the parameters. Under suitable conditions, we prove the asymptotic convergence of the blockwise waveform relaxation method for block $H$-matrix of different types. Moreover, we demonstrate the monotone convergence properties as well as the monotone comparison

* Received May 16, 2002.
1) Subsidized by The Special Funds For Major State Basic Research Projects G1999032803.

theorems, which reveal the influence of the matrix splitting and the initial approximation upon the convergence rate of this kind of method.

The organization of this paper is as follows. We introduce the definition of block $H$-matrix and some related properties in Section 2, and establish the blockwise waveform relaxation method in Section 3. The asymptotic and monotone convergence properties of the blockwise waveform relaxation method are discussed in Sections 4 and 5, respectively. In Section 6, we demonstrate the comparison theorem for the waveform relaxation methods. As a consequence, the result of the convergence rates between the pointwise and blockwise waveform accelerated overrelaxation methods is given in the monotone sense. We present numerical results by solving a two-dimensional heat equation in Section 7, and at last, we end this paper with a brief concluding remark in Section 8.

## 2. Preliminaries

The partial orderings " $\leq$ ", " $<$ " and the absolute value $|\bullet|$ in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ are defined according to the elements. For a matrix $A\in\mathbb{C}^{n\times n}$, let $\ell(\leq n)$ and $n_i(\leq n)(i=1,\ldots,\ell)$ be positive integers satisfying $\sum_{i=1}^{\ell} n_i=n$, and define the blockwise vector and matrix spaces[3]

$$\begin{aligned}
\mathbb{V}_n(n_1,\ldots,n_\ell) &= \{\ x\in\mathbb{C}^n\ |\ x=(x_1^T,\ldots,x_\ell^T)^T,\ x_i\in\mathbb{C}^{n_i},\ i=1,\ldots,\ell\ \};\\
\mathbb{L}_n(n_1,\ldots,n_\ell) &= \{\ A\in\mathbb{C}^{n\times n}\ |\ A=(A_{ij}),\ A_{ij}\in\mathbb{C}^{n_i\times n_j},\ i,j=1,\ldots,\ell\ \};\\
\mathbb{L}_{n,I}(n_1,\ldots,n_\ell) &= \{M=(M_{ij})\in\mathbb{L}_n(n_1,\ldots,n_\ell)|M_{ii}\in\mathbb{C}^{n_i\times n_i}\ \text{nonsingular},\ i=1,\ldots,\ell\},
\end{aligned}$$

which will be denoted simply by $\mathbb{V}_n,\mathbb{L}_n$ and $\mathbb{L}_{n,I}$, respectively, if there is no ambiguity.

A matrix $G=(g_{ij})\in\mathbb{R}^{n\times n}$ is called an $M$-matrix if $g_{ij}\leq 0(i\neq j),i,j=1,\ldots,n$, and $G^{-1}$ exists with $G^{-1}\geq 0$; an $H$-matrix if its comparison matrix $\mathfrak{M}(G)$ is an $M$-matrix, where $\mathfrak{M}(G)=(\mathfrak{m}_{ij})$ is an $n\times n$ matrix with $\mathfrak{m}_{ii}=|g_{ii}|$ and $\mathfrak{m}_{ij}=-|g_{ij}|(i\neq j)$; and an $H_+$-matrix if $G$ is an $H$-matrix satisfying $g_{ii}>0(i=1,\ldots,n)$[8]. Evidently, if we denote

$$D_G=diag(g_{11},\ldots,g_{nn}),\quad B_G=D_G-G,\quad J_G=D_G^{-1}B_G,$$

and

$$\mathcal{L}^{n\times n}=\{M=(m_{ij})\ |\ M\in\mathbb{R}^{n\times n},\ m_{ij}\leq 0,\ i\neq j,\ i,j=1,\ldots,n\ \},$$

then $G\in\mathcal{L}^{n\times n}$ with positive diagonals is an $M$-matrix if and only if $\rho(J_G)<1$[29], where $\rho(\bullet)$ denotes the spectral radius of a matrix. For $M\in\mathbb{L}_{n,I}$, its type-$I$ (type-$II$) block comparison matrix $\langle M\rangle=(\langle M\rangle_{ij})\in\mathbb{R}^{\ell\times\ell}$ ($\langle\langle M\rangle\rangle=(\langle\langle M\rangle\rangle_{ij})\in\mathbb{R}^{\ell\times\ell}$) is defined by $\langle M\rangle_{ii}=\|M_{ii}^{-1}\|^{-1}$ ($\langle\langle M\rangle\rangle_{ii}=1$) and $\langle M\rangle_{ij}=-\|M_{ij}\|$ ($\langle\langle M\rangle\rangle_{ij}=-\|M_{ii}^{-1}M_{ij}\|$) for $i\neq j,\ i,j=1,\ldots,\ell$; see [6, 7, 9, 15, 25]. $M\in\mathbb{L}_{n,I}$ is called a type-$I$ (type-$II$) block $H$-matrix if $\langle M\rangle(\langle\langle M\rangle\rangle)$ is an $M$-matrix, and we simply denote it by $M\in H_B^{(I)}(M\in H_B^{(II)})$. Evidently, it holds that $H_B^{(I)}\subseteq H_B^{(II)}$.

For $M\in\mathbb{L}_n$, we use $[M]=(\|M_{ij}\|)\in\mathbb{R}^{\ell\times\ell}$ to represent the block absolute value. The block absolute value of a vector $x\in\mathbb{V}_n$ can be defined in an analogous way.

The following lemmas will be frequently used in what follows.

**Lemma 2.1.** *Let $L,M\in\mathbb{L}_n,x,y\in\mathbb{V}_n$ and $\gamma\in\mathbb{C}^1$. Then*

*(1)* $|[L]-[M]|\leq[L+M]\leq[L]+[M]$ $(|[x]-[y]|\leq[x+y]\leq[x]+[y])$; [3]

*(2)* $[LM]\leq[L][M]$ $([Mx]\leq[M][x])$; [3]

*(3)* $[\gamma M]=|\gamma|[M]$ $([\gamma x]=|\gamma|[x])$; [3]

*(4)* $\rho(M)\leq\rho([M])$.