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THE MECHANICAL QUADRATURE METHODS AND THEIR EXTRAPOLATION FOR SOLVING BIE OF STEKLOV EIGENVALUE PROBLEMS *1)

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Abstract

By means of the potential theory Steklov eigenvalue problems are transformed into general eigenvalue problems of boundary integral equations (BIE) with the logarithmic singularity.Using the quadrature rules[1], the paper presents quadrature methods for BIE of Steklov eigenvalue problem, which possess high accuracies $O(h^3)$ and low computing complexities. Moreover, an asymptotic expansion of the errors with odd powers is shown. Using h^3- Richardson extrapolation, we can not only improve the accuracy order of approximations, but also derive a posterior estimate as adaptive algorithms. The efficiency of the algorithm is illustrated by some examples.

Mathematics subject classification: 65N25.
Key words: Steklov eigenvalue problem, Boundary integral equation, Quadrature method, Richardson extrapolation.

1. Introduction

Consider the following Steklov eigenvalue problem:

$$\begin{cases} \Delta\tilde{u} - \alpha^2\tilde{u} = 0, \text{ in } \Omega, \\ \frac{\partial\tilde{u}}{\partial n} = \lambda\tilde{u}, \text{ on } \Gamma, \end{cases} \tag{1.1}$$

where α is a constant. Ω is a bounded domain with the boundary Γ, and $\frac{\partial}{\partial n}$ is the outward normal derivative on Γ. This problem arises from many applications, e.g., the free membrane and heat flow problems.

Courant and Hilbert[5] studied the problem (1.1). Bramble and Osborn[3] gave finite element method and its error estimate for solving the equation (1.1). Liu and Ortiz[9] gave finite difference methods and Tao-Method. Obviously, the problem (1.1) is easily converted into the eigenvalue problem of the boundary integral equation, so the boundary element method(BEM) solved (1.1) is more advantageous. Using a new variational formula of BIE, Han, Guan and He[6,7], and Tang, Guan and Han[13] derived a new BIE of the problem (1.1) and obtained its approximate BEM, which can keep the self adjoint property of the original problem. Although their approximate methods are very efficient, however, there exist two disadvantages: (1) each element of the discrete matrix as full has to calculate a double improper integral; (2) the order of accuracy only is $O(h^2)$. In [10,11], the mechanical quadrature methods for solving the boundary integral equations are constructed, where the convergence and asymptotic expansion with h^3 are proved by the collectively compact and asymptotically compact theory[1,2,4].

In this paper Steklov eigenvalue problem will be transformed into a general eigenvalue problem of BIE with the logarithmic singularity. Applying Side's quadrature rules[12], we present the mechanical quadrature methods of BIE for solving Steklov eigenvalue problem,

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in which the generation of the discrete matrix is without any calculations of integrals. Since the asymptotic expansions of the errors with the power h^3 are shown, our methods imply that using h^3-Richardson extrapolation not only improves the accuracy order of approximations, but also a posteriori estimate as adaptive algorithm is got. The numerical examples show that our methods have the accuracy order $O(h^3)$. The extrapolation and a posteriori estimate are very effective.

By the potential theory, the problem (1.1) is easily transformed into the following eigenvalue problem of BIE

$$\frac{1}{2}\tilde{u}\,(x) - \int_{\Gamma} \tilde{u}(y)\frac{\partial}{\partial n_y}\Phi(x,y)ds_y = -\lambda \int_{\Gamma} \tilde{u}(y)\Phi(x,y)ds_y, x \in \Gamma, \tag{1.2}$$

where $x = (x_1, x_2)\,,\, y = (y_1, y_2),\, |x-y|^2 = (x_1-y_1)^2 + (x_2-y_2)^2$,

$$\Phi(x,y) = \begin{cases} -K_0(\alpha|x-y|)/(2\pi), \text{ for } \alpha > 0 \\ (2\pi)^{-1}\log|x-y|, \text{ for } \alpha = 0 \end{cases}$$

is the fundamental solution of equation (1.1); $K_0\,(z)$ is the modified Bessel function with

$$K_0\,(z) \approx -\log z + \log 2 - \gamma,\ z \to 0;$$

and $\gamma \approx 0.5772$ is Euler constant.

2. Mechanical Quadrature Methods

Assume that Γ is a smooth closed curve described by the parameter mapping $x(t) = (x_1(t), x_2(t))$ with $(x_1^{'}(t))^2 + (x_2^{'}(t))^2 > 0$. Let $C^m[0,2\pi]$ denote the set of m times differentiable periodic functions with periodic 2π. Define the following integral operators on $C^m[0,2\pi]$

$$(Ku)(s) = \int_0^{2\pi} k\,(t,s)\,u(t)dt, \tag{2.1}$$

and

$$(Bu)(s) = \int_0^{2\pi} b(t,s)u\,(t)\,dt, \tag{2.2}$$

where $u(t) = \tilde{u}(x_1(t), x_2(t))$. $k\,(t,s) = \frac{\partial}{\partial n_y}(\Phi(x(t), x(s)))|x^{'}(t)|/\pi$ is smooth function; however, $b(t,s) = \Phi(x(t), x(s))|x^{'}(t)|/\pi$ is with the logarithmic singularity. The equation (1.2) is equivalent to the operator eigenvalue problem

$$\begin{cases} (I-K)u = -\lambda Bu, \\ ||u||_{0,\Gamma}^2 = \int_0^{2\pi} u^2(s)ds = 1. \end{cases} \tag{2.3}$$

Take a mesh width $h = 2\pi/n$, and $t_j = s_j = jh$, $j = 0, \cdots, n-1$. By the trapezoidal and quadrature rule[12] we construct the approximate operators of K and B

$$\begin{cases} (K_h u)(s) = h\sum_{j=0}^{n-1} k(t_j, s)u(t_j), \\ (B_h u)(s) = h\sum_{j=0}^{n-1} b_n(t_j, s)u(t_j), \end{cases} \tag{2.4}$$

where

$$b_n(t,s) = \begin{cases} b(t,s), \text{ for } |t-s| \geq h; \\ h[\log\frac{h}{2\pi} + \log|x^{'}\,(s)\,| + \varepsilon_\alpha]|x'\,(s)\,|, \text{ for } |t-s| < h \end{cases}$$

is a continuous approximation of $b(t,s)$ and

$$\varepsilon_\alpha = \begin{cases} -\log(2\alpha) - \gamma, \text{ for } \alpha > 0; \\ 0, \text{for } \alpha = 0. \end{cases}$$