Journal of Computational Mathematics, Vol.22, No.5, 2004, 727–734.

$\mathcal{H}$-STABILITY OF RUNGE-KUTTA METHODS WITH VARIABLE STEPSIZE FOR SYSTEM OF PANTOGRAPH EQUATIONS *

Yang Xu Jing-jun Zhao Ming-zhu Liu
(*Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China*)

Abstract

This paper deals with $\mathcal{H}$-stability of Runge-Kutta methods with variable stepsize for the system of pantograph equations. It is shown that both Runge-Kutta methods with nonsingular matrix coefficient A and stiffly accurate Runge-Kutta methods are $\mathcal{H}$-stable if and only if the modulus of stability function at infinity is less than 1.

Mathematics subject classification: 65L06, 65L20.
Key words: Delay differential equations, Stability, Runge-Kutta method.

1. Introduction

In recent years, much research has been focused on theoretical and numerical solutions of pantograph equations. These systems can be found in variety of scientific and engineering fields such as analytic number theory, nonlinear dynamical systems, collection of current by the pantograph of an electric locomotive and so on, which have a comprehensive list in [7].

As far as we know the delay differential equations can be classified into two cases according to time lag, one is those with finite time lags, and the other is those with infinite time lags. There are remarkable differences, both analytically and numerically, between these two classes. Theoretical study of the second class of equations can be found in [7]. The numerical methods for this class have been studied by [2], in which the grid is uniform. However, this kind of equation has unbounded time lags, it is usually difficult to investigate numerically the long time dynamical behavior of exact solution due to limited computer memory as shown in [10]. There are two kinds of ways to avoid the storage problem. One is to transform the equation into an equation with constant time lag and variable coefficients as shown in [9] and apply a numerical method with constant stepsize $h = \frac{-\ln q}{m}$ to it. As a matter of fact, it seems like applying the numerical method with a grid which is not uniform, $t_n = q^{-\frac{n}{m}}$ and variable stepsizes $h_n = q^{-\frac{n}{m}}(q^{-\frac{1}{m}} - 1)$. Another way is applying a numerical method with variable stepsizes to the equation directly, which are considered in [1,10]. The nonconstant stepsize strategy is considered in [1], which is a special case of that introduced by Liu [10].

In this paper, we focus on the $\mathcal{H}$-stability of Runge-Kutta methods with variable stepsize applied to the system of pantograph equations. Some conclusions about the asymptotical stability of analytical solutions to the system are recalled. Furthermore, variable stepsize scheme is given. Finally, Runge-Kutta methods with nonsingular matrix coefficient A and stiffly accurate Runge-Kutta methods are applied to this system, respectively. The same sufficient and necessary condition such that the methods are $\mathcal{H}$-stable is presented.

2. Runge-Kutta Method with Variable Stepsize

In this paper, we consider the two-dimensional pantograph equations:

$$\begin{cases} x'(t) &= \lambda_1 x(t) + \mu_1 y(q_1 t), \\ y'(t) &= \lambda_2 y(t) + \mu_2 x(q_2 t), \end{cases} \qquad t > 0, \tag{2.1}$$

* Received July 22, 2002

where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathcal{C}$, $0 < q_2 \le q_1 < 1$ with the initial conditions $x(0) = x_0$, $y(0) = y_0$.

The existence and uniqueness of solutions to the system (2.1) has been studied by [7]. It is demonstrated in our another recent paper [11] that the solutions tend to zero (algebraically) if

$$|\mu_1| < -\Re\lambda_1, \quad |\mu_2| < -\Re\lambda_2 \quad \text{and} \quad \Re\lambda_1 < 0, \quad \Re\lambda_2 < 0, \tag{2.2}$$

where $\Re\lambda$ is the real part of λ.

Definition 2.1. *System (2.1) is called asymptotically stable if, for any $q_1, q_2 \in (0,1)$ and any initial values, the solutions $x(t), y(t)$ of this system tend to zero as t approaches infinity.*

Thus the analytical stability region can be defined as $\mathbf{S}=\{(\lambda_1,\lambda_2,\mu_1,\mu_2) \in \mathcal{C}^4 | (\lambda_1,\lambda_2,\mu_1,\mu_2)$ satisfies condition (2.2)$\}$.

In the next two sections, we focus our attention on the numerically asymptotical stability of Runge-Kutta method. Now we recall the method presented in [3] and the variable stepsize schemes introduced in [11].

For the general pantograph equation:

$$z'(t) = f(t, z(t), z(qt)),$$

where $0 < q < 1$.

The Runge-Kutta method, presented by [3], gives out the recurrence relation:

$$\begin{cases} z_{n+1} &= z_n + h_{n+1} \sum\limits_{i=1}^{s} b_i f(t_n + c_i h_{n+1}, Z_{n,i}, z^h(t_n + c_i h_{n+1})), \\ Z_{n,i} &= z_n + h_{n+1} \sum\limits_{j=1}^{s} a_{ij} f(t_n + c_j h_{n+1}, Z_{n,j}, z^h(t_n + c_j h_{n+1})), \end{cases} \tag{2.3}$$

where (A, b, c) denotes Runge-Kutta method, with matrix $A = (a_{ij})_{s\times s}$, vectors $b = (b_1, b_2, \ldots, b_s)^T$, $c = (c_1, c_2, \ldots, c_s)^T$. And $Z_{n,i}$, $z^h(t_n + c_i h_{n+1})$ can be interpreted as the approximation to $z(t_n + c_i h_{n+1})$ and $z(q(t_n + c_i h_{n+1}))$ respectively, for $i = 1, \ldots, s$.

Here, $z^h(t)$ is defined by the piecewise linear interpolation for $t > 0$, i.e.,

$$z^h(t) = \frac{t - t_i}{t_j - t_i} z_j + \frac{t_j - t}{t_j - t_i} z_i, \qquad t_i < t \le t_j.$$

Here, variable stepsize introduced in [11] is recalled.

For simplicity, without loss of generality, we assume that $t_0 = 1$ and the numerical solution is available till some point $t_0 > 0$.

Let $T_l = \frac{1}{q^l} t_0 = \frac{1}{q^l}$, $T_{l+1} = \frac{1}{q^{l+1}} t_0 = \frac{1}{q^{l+1}}$, where $l = 0, 1, 2, \ldots$.

If we are interested in the values of $z(t)$ at points $t^{(1)}, t^{(2)}, \ldots, t^{(m-1)}$ with $t^{(1)} < t^{(2)} < \ldots < t^{(m-1)}$, then there must exist integers $k_1, k_2, \ldots, k_{m-1}$ such that $q^{k_i} \cdot t^{(i)} \in [T_0, T_1)$, for $i = 1, 2, \ldots, m-1$ and $q \in (0,1)$.

We define the grid points and variable stepsizes as follows,

$$t_0 = T_0, \quad t_m = T_1, \quad t_i = q^{k_i} t^{(i)},$$

and

$$t_{km+i} = q^{-k} t_i, \quad h_{n+1} = t_{n+1} - t_n,$$

for $k = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, m-1$.