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# THE PREDICTION-CORRECTION LEGENDRE COLLOCATION METHOD FOR NONLINEAR EVOLUTIONARY PROBLEMS *1)

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**Abstract**

The initial-boundary value problem of Burgers equation is considered. A prediction-correction Legendre collocation scheme is presented, which is easy to be performed. Its numerical solution possesses the accuracy of second-order in time and higher order in space. Numerical results are reported, which show the high accuracy of this approach. The techniques used in this paper are also applicable to other nonlinear evolutionary problems.

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*Key words*: Burgers equation, Prediction-correction operator, Legendre collocation approximation, Convergence, Numerical results.

## 1. Introduction

Since spectral methods possess the accuracy of "infinite" order, they have been widely applied to computational fluid dynamics, e.g., see [1-3]. As we know, the Burgers equation plays an important role in fluid dynamics. Maday and Quarteroni [4] studied Legendre and Chebyshev spectral approximations to the steady problem. Recently Bialecki and Karageorghis [10] considered Legendre collocation method and He Li-ping and Sun Shun-kai [11] provided a fast direct Legendre collocation algorithm for linear elliptic problems. Ma He-ping and Guo Ben-yu [6] developed Chebyshev spectral methods for unsteady problem. In some existing work, the temporal discretizations for nonlinear evolutionary problems is of the first order, and so limits the merit of spectral approximations in space, e.g., see [7]. To remedy this trouble, a prediction-correction Legendre spectral scheme was proposed in [8,9], which produces precise numerical results, in which the linear leading terms are approximated implicitely and the nonlinear terms are approximated explicitely at each step. Thus we can solve it explicitely by using the algorithm in [11]. However, how to analyze the high accuracy for such an approach is still an open problem. Indeed, so far, there has been no result on error estimate of prediction-correction operator Legendre collocation method. Since we can not derive an explicit relationship between the numerical solution and its predicted one, and so the analysis is very difficult.

In this paper, we take the unsteady Burgers equation as an example to show how to construct a reasonable Legendre spectral collocation approximation using prediction-correction operator in time and how to analyze the errors. Let $T>0$, $\Lambda=(-1,1)$, $\partial\Lambda=\{-1,1\}$ and $\mu>0$ be the kinetic viscosity. $f(x)$ and $u_0$ describe the source term and the initial state. Then the unsteady Burgers equation is of the form

$$\begin{cases} \dfrac{\partial u}{\partial t}-\mu\dfrac{\partial^2 u}{\partial x^2}+\dfrac{1}{2}\dfrac{\partial}{\partial x}u^2=f, & \text{in } \ \Lambda\times(0,T], \\ u=0, & \text{on } \ \partial\Lambda\times(0,T], \\ u(x,0)=u_0(x), & \text{in } \ \Lambda\cup\partial\Lambda. \end{cases} \tag{1.1}$$

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In Section 2, we construct the scheme and present the convergence. In Section 3, we present the numerical results which show the high accuracy of this method. We list some lemmas in Section 4 and prove the accuracy of second-order in time and high order in space in the final section. The technique provided in this paper are also applicable to other nonlinear evolutionary problems.

## 2. The Prediction-correction Legendre Collocation Scheme

Throughout the paper we use Sobolev spaces $H^r(\Lambda)$ and $H_0^r(\Lambda)$. For simplicity, let $L^2(\Lambda) = H^0(\Lambda)$. Their definitions and properties can be found in [12]. The inner product, the semi-norm and the norm of $H^r(\Lambda)$, $r \geq 0$ are denoted by $(\cdot,\cdot)_r$, $|\cdot|_r$, $\|\cdot\|_r$ respectively. If $r=0$, then the index $r$ is omitted. We recall that the usual semi-norm $|\cdot|_r$ is equivalent to the norm $\|\cdot\|_r$ in $H_0^r(\Lambda)$. Further let $H^{-r}(\Lambda)$ be the dual space of $H_0^r(\Lambda)$, and $<\cdot,\cdot>_{L(H^{-r},H_0^r)}$ be the duality parting between $H^{-r}(\Lambda)$ and $H_0^r(\Lambda)$. Define the bilinear form $A(\cdot,\cdot): H^1(\Lambda)\times H^1(\Lambda) \longmapsto R$ and the trilinear form $B(\cdot,\cdot,\cdot): L^4(\Lambda)\times L^4(\Lambda)\times H^1(\Lambda)\longmapsto R$ as follows

$$\begin{aligned} A(u,v) &= (\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}), \quad \forall u, \quad v \in H^1(\Lambda), \\ B(u,v,w) &= -\frac{1}{2}(uv, \frac{\partial w}{\partial x}), \quad \forall u,v \in L^4(\Lambda), \quad w \in H^1(\Lambda). \end{aligned} \tag{2.1}$$

So the weak formulation of (1.1) is to find a function $u \in L^2(0,T;H_0^1(\Lambda))\bigcap L^\infty(0,T,\ L^2(\Lambda))$ such that

$$\begin{cases} (\frac{\partial u}{\partial t}, v) + \mu A(u,v) + B(u,u,v) = < f, v >_{L(H^{-1},H_0^1)}, \quad \forall v \in H_0^1(\Lambda). \\ u(x,0) = u_0(x). \end{cases} \tag{2.2}$$

It can be proved that if $f \in L^2(0,T;H^{-1}(\Lambda))$ and $u_0 \in L^2(\Lambda)$, then (2.2) has a unique solution.

Let $P_N(\Lambda)$ be the set of all algebraic polynomials of degree at most $N$ and $P_N^0(\Lambda) = P_N(\Lambda)\bigcap H_0^1(\Lambda)$. We define the orthogonal projection operator $P_N^1: H_0^1(\Lambda)\longmapsto P_N^0(\Lambda)$ such that

$$A(u - P_N^1 u, v_N) = 0, \quad \forall v_N \in P_N^0(\Lambda).$$

Also, we shall use the orthogonal projection operator $P_N: L^2(\Lambda)\longmapsto P_N(\Lambda)$ defined as

$$(u - P_N u, v_N) = 0, \quad \forall v_N \in P_N(\Lambda).$$

Denote by $\{x_j\}_{j=0,1,\cdots,N}$ and $\{\omega_j\}_{j=0,1,\cdots,N}$ the nodes and weights of the Gauss-Lobatto-Legendre quadratuare formula on $\overline{\Lambda}$=[-1,1]. Let $I_N: C(\overline{\Lambda})\longmapsto P_N(\Lambda)$ be the interpolation operator on $\{x_j\}_{j=0,1,\cdots,N}$. It is obvious that

$$I_N u(x_j) = u(x_j), \quad j = 0,1,\cdots,N.$$

The discrete inner product and norm are defined as

$$(u,v)_N = \sum_{j=0}^{N} u(x_j)v(x_j)\omega_j, \| u \|_N = (u,u)_N^{\frac{1}{2}}.$$

Also, we shall use the norm

$$\| u \|_{L^q(\Lambda),N} = \begin{cases} (\sum_{j=0}^{N} | u(x_j) |^q \omega_j)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \max_{0\leq j\leq N} | u(x_j) |, & q = \infty. \end{cases}$$