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THE STRUCTURAL CHARACTERIZATION AND LOCALLY SUPPORTED BASES FOR BIVARIATE SUPER SPLINES *1)

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Abstract

Super splines are bivariate splines defined on triangulations, where the smoothness enforced at the vertices is larger than the smoothness enforced across the edges. In this paper, the smoothness conditions and conformality conditions for super splines are presented. Three locally supported super splines on type-1 triangulation are presented. Moreover, the criteria to select local bases is also given. By using local supported super spline function, a variation-diminishing operator is built. The approximation properties of the operator are also presented.

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1. Introduction

Let D be a polygonal domain in R^2 and Δ a triangulation of D consisting of finitely straight lines or line segments defined by $\Gamma_i : y - a_i x - b_i = 0, i = 1, \cdots, N$. Denote by $v_i, i = 1, \cdots, V_I$ all the vertices of Δ. Denote by $D_i, i = 1, \cdots, T$, all the cells of Δ. For integers $k \geq \rho \geq r \geq 0$, we say that

$$S_k^{r,\rho}(\Delta) = \{s \in S_k^r(\Delta) : s \in C^\rho(v_i), i = 1, \cdots, V_I\},$$

is a *super spline space of degree k and smoothness r, ρ* (cf.[5,13,12]), where $C^\rho(v)$ denotes the set of functions defined on D which are ρ times continuously differentiable at the point v and $S_k^r(\Delta)$ is an ordinary spline space defined as

$$S_k^r(\Delta) = \{s \in C^r(\Omega) : s|_{D_i} \in \mathbf{P}_k(x,y) \forall i\}.$$

Throughout the paper, $\mathbf{P}_k(x,y)$ and $\mathbf{P}_k(x)$ denote the collection of polynomials

$$\mathbf{P}_k(x,y) := \{\sum_{i=0}^{k}\sum_{j=0}^{k-i} c_{ij} x^i y^j | c_{ij} \in R\}, \mathbf{P}_k(x) := \{\sum_{i=0}^{k} c_i x^i | c_i \in R\},$$

respectively. Moreover, if $k < 0$, $\mathbf{P}_k(x,y)$ and $\mathbf{P}_k(x)$ are both equal to zero. If $S_k^{r,\rho}(\Delta) \neq S_k^\rho(\Delta)$, the super spline space $S_k^{r,\rho}(\Delta)$ is called a *nontrivial super spline space of degree k and smoothness r, ρ.* Super splines have strongly applied background in finite elements, vertex spline and Hermite interpolation. In [13], the relation between super spline theory and finite element theory was introduced. In [10,2,11], by using super spline, bivariate macro element was built. In [9], based on super spline spaces, Hermite interpolation was discussed. In [1,4,13,8,12], the

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dimensions of $S_k^{r,\rho}(\Delta)$ were given. In these papers, super splines were discussed by a B net method. In this paper, the smooth cofactors method for studying super splines is presented. The method is more effective for solving some problems about super splines.

By using Bezout's theorem from algebraic geometry, Wang discovered the following smooth conditions and the conformality conditions for bivariate splines (cf.[15]).

Theorem 1[15]**.** *The function $s(x,y)$ is a bivariate spline belonging to $S_k^{\mu}(\Delta)$ if and only if the following conditions are satisfied.*

(i) For any grid-segment Γ_i defined by $l_i(x,y)=0$, there exists the so-called smoothing cofactor $q_i(x,y)$ such that

$$p_{i1}(x,y)-p_{i2}(x,y)=l_i^{\mu+1}(x,y)q_i(x,y), \tag{1}$$

where the polynomials p_{i1} and p_{i2} are determined by the restriction of $s(x,y)$ to the two cells D_{i1} and D_{i2} with Γ_i as common edge and $q_i \in \mathbf{P}_{k-(\mu+1)}(x,y)$.

(ii) For any interior vertex v_j of Δ, the following conformality conditions are satisfied

$$\sum_i (l_i^{(j)}(x,y))^{\mu+1} q_i^{(j)}(x,y) \equiv 0, \tag{2}$$

where the summation is taken over all the interior edges $\Gamma_i^{(j)}$ passing through v_j and the signs of the smoothing cofactors $q_i^{(j)}$ are refixed in such a way that when a point crosses $\Gamma_i^{(j)}$ from D_{i2} to D_{i1} it goes around v_j in a counter-clockwise manner.

Smooth conditions and the conformality conditions are very effective tools for studying bivariate splines (cf.[16]). The purpose of this paper is to present the smooth conditions and the conformality conditions for super splines. Using the smooth conditions and conformality conditions , the locally supported bases of super splines on type-1 triangulation are also discussed. The local supported bases of super splines have a wide range of applications in approximation, interpolation, numerical analysis and finite element methods. We shall only discuss some approximation properties arising from the variation-diminishing super spline series.

2. The Smooth Conditions and the Conformality Conditions for Super Spline

To obtain the smooth conditions and the conformality conditions for super splines, we firstly introduce a lemma. One can find a similar result in [14].

Lemma 1. *Denote by $l(x,y)$ the straight line $y-ax-b=0$. Let $p(x,y)\in \mathbf{P}_k(x,y)$ and $(x_1,y_1),(x_2,y_2)$ be two distinct points lying on l. Then $\frac{\partial^n p(x,y)}{\partial^j x \partial^{n-j} y}|_{(x_1,y_1)}=0$, $\frac{\partial^n p(x,y)}{\partial^j x \partial^{n-j} y}|_{(x_2,y_2)}=0, j\le n\le \mu$, if and only if there exist $q(x,y)\in \mathbf{P}_{k-\mu-1}(x,y)$ and $c_m(x)\in \mathbf{P}_{k-\mu-m-1}(x)$ such that*

$$p(x,y)=(y-ax-b)^{\mu+1}q(x,y)+\sum_{m=1}^{\mu+1}(y-ax-b)^{\mu+1-m}(\prod_{i=1}^{2}(x-x_i))^m c_m(x) \tag{3}$$

where $c_m(x)\equiv 0$ provided $k-\mu-m-1<0$.

Proof. There exist $q(x,y)\in \mathbf{P}_{k-1}(x,y)$ and $c(x)\in \mathbf{P}_k(x)$, such that

$$p(x,y)=(y-ax-b)q(x,y)+c(x). \tag{4}$$

When $\mu=0$, then $c(x_1)=0, c(x_2)=0$, i.e., there exists $c_1(x)$ such that $c(x)=(x-x_1)(x-x_2)c_1(x)$. So, the theorem holds for $\mu=0$. Suppose that the lemma holds for $\mu=g-1$, i.e. there exist $q_0(x,y)$ and $c_m^{(0)}(x)$ such that

$$p(x,y)=(y-ax-b)^g q_0(x,y)+\sum_{m=1}^{g}(y-ax-b)^{g-m}(\prod_{i=1}^{2}(x-x_i))^m c_m^{(0)}(x). \tag{5}$$