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# CONVERGENCE DOMAINS OF AOR TYPE ITERATIVE MATRICES FOR SOLVING NON-HERMITIAN LINEAR SYSTEMS *1)

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**Abstract**

We discuss AOR type iterative methods for solving non-Hermitian linear systems based on Hermitian splitting and skew-Hermitian splitting. Convergence domains of iterative matrices are given and optimal parameters are investigated for skew-Hermitian splitting. Numerical examples are presented to compare the effectiveness of the iterative methods in different points in the domain. In addition, a model problem of three-dimensional convection-diffusion equation is used to illustrated the application of our results.

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*Key words*: Convergence, AOR type iterative methods, Skew-Hermitian splitting, Hermitian splitting, Linear systems.

## 1. Introduction

Given a nonsingular system of linear equations

$$Ax = b, \quad A \in \mathbb{C}^{n\times n}, \quad b \in \mathbb{C}^n, \tag{1.1}$$

where the coefficient matrix $A$ is non-Hermitian, we assume that $D = diag(A)$ is nonsingular. Since both splittings $A = M - N$ and $D^{-1}A = D^{-1}M - D^{-1}N$ lead to the same iteration operator, we may assume, without loss of generality, that

$$A = I - B, \quad \text{where} \quad diag(B) = 0. \tag{1.2}$$

It is convenient to regard any splitting $M - N$ of $A = I - B$ as having the identity incorporated into $M$, and we thus write

$$M = I - M_B, \quad \text{and} \quad N = B - M_B. \tag{1.3}$$

Then, with AOR type iteration matrix [4]

$$T_{\omega,\gamma} = (I - \gamma M_B)^{-1}\{(1-\omega)I + (\omega-\gamma)M_B + \omega N\}, \tag{1.4}$$

and $c_{\omega,\gamma} = \omega(I - \gamma M_B)^{-1}b$, we have the associated AOR type iterative method [1, 3]

$$x^{(i+1)} = T_{\omega,\gamma}x^{(i)} + c_{\omega,\gamma} \quad i = 1, 2, \ldots.$$

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Letting

$$F = \frac{B + B^*}{2} \quad \text{and} \quad G = \frac{B - B^*}{2}$$

denote, respectively, the Hermitian and skew-Hermitian parts of $B$, then the Hermitian splitting of $A$ is defined by [4]

$$A = M^h - N^h \quad \text{with} \quad M^h = I - F \quad \text{and} \quad N^h = G \tag{1.5}$$

where we assume that $M^h$ is invertible, which is, for instance, guaranteed if the Hermitian part $M^h$ of $A$ is positive definite. The associated skew-Hermitian splitting of $A$ is given by [4]

$$A = M^s - N^s \quad \text{with} \quad M^s = I - G \quad \text{and} \quad N^s = F. \tag{1.6}$$

In this way, the specific splitting defined in (1.5) and (1.6) generates the following two AOR type iterative methods

$$x^{(m)} = T^h_{\omega,\gamma} x^{(m-1)} + c^h_{\omega,\gamma} \quad (m = 1, 2, \cdots), \tag{1.7}$$

where

$$T^h_{\omega,\gamma} = (I - \gamma F)^{-1}\{(1-\omega)I + (\omega - \gamma)F + \omega G\}, \quad c^h_{\omega,\gamma} = \omega(I - \gamma F)^{-1} b,$$

and

$$x^{(m)} = T^s_{\omega,\gamma} x^{(m-1)} + c^s_{\omega,\gamma} \quad (m = 1, 2, \cdots), \tag{1.8}$$

where

$$T^s_{\omega,\gamma} = (I - \gamma G)^{-1}\{(1-\omega)I + (\omega - \gamma)G + \omega F\}, \quad c^s_{\omega,\gamma} = \omega(I - \gamma G)^{-1} b,$$

Each of these methods depends on two parameters $\gamma$ and $\omega$.

The last fourty years have produced many methods for solving linear systems. Much is known in the literature [7, 8] about basic ones. AOR type method which was proposed by A Hadjidimos in [3] in 1978 is a accelerated overrelaxation method. Using Hermitian and skew-Hermitian matrix splitting and combining with krylov subspace iterative methods, many methods have been developed [2, 5]. In [1] Bai has also given the convergence domain of the matrix multisplitting relaxation methods. Based on the technique in [4], further discuss of the convergence of AOR type methods will be given in our paper.

The organization of this paper is as follows. In section 2, we study the convergence properties of AOR type iterative methods for Hermitian splitting and skew-Hermitian splitting and give the near optimal parameters for skew-Hermitian splitting. In section 3, the three-dimensional convection-diffusion equation is employed as a model problem to illustrate the application of our results. Numerical experiments are presented in section 4 to compare the effectiveness of our methods in different points of convergence domains.

## 2. Convergence of AOR type Iterative Methods

**Lemma 2.1.** *If $I - \gamma M_B$ is nonsingular and if $\tau$ is a eigenvalue of $T_{\omega,\gamma}$ of (1.4) with eigenvector $v$, normalized by $v^* v = 1$, then*

$$\tau = \frac{1 - \omega + (\omega - \gamma)m + \omega\eta}{1 - \gamma m}, \quad \textit{where} \quad \eta = v^* N v \quad \textit{and} \quad m = v^* M_B v. \tag{2.1}$$