Journal of Computational Mathematics, Vol.22, No.6, 2004, 833–856.

COMBINATIVE PRECONDITIONERS OF MODIFIED INCOMPLETE CHOLESKY FACTORIZATION AND SHERMAN-MORRISON-WOODBURY UPDATE FOR SELF-ADJOINT ELLIPTIC DIRICHLET-PERIODIC BOUNDARY VALUE PROBLEMS *1)

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Abstract

For the system of linear equations arising from discretization of the second-order self-adjoint elliptic Dirichlet-periodic boundary value problems, by making use of the special structure of the coefficient matrix we present a class of combinative preconditioners which are technical combinations of modified incomplete Cholesky factorizations and Sherman-Morrison-Woodbury update. Theoretical analyses show that the condition numbers of the preconditioned matrices can be reduced to $\mathcal{O}(h^{-1})$, one order smaller than the condition number $\mathcal{O}(h^{-2})$ of the original matrix. Numerical implementations show that the resulting preconditioned conjugate gradient methods are feasible, robust and efficient for solving this class of linear systems.

Mathematics subject classification: 65F10, 65F50.
Key words: System of linear equations, Conjugate gradient method, Incomplete Cholesky factorization, Sherman-Morrison-Woodbury formula, Conditioning.

1. Introduction

Consider the two-dimensional second-order self-adjoint elliptic partial differential equation

$$-\nabla \cdot (a(\xi,\eta)\cdot \nabla u) + \theta(\xi,\eta)\cdot u = f(\xi,\eta) \tag{1.1}$$

in the unit square $\Omega = (0,1)\times(0,1)$ with the boundary conditions

$$\begin{cases} u(0,\eta) &= g_0^{(1)}(\eta), \qquad u(1,\eta) = g_1^{(1)}(\eta), \\ u(\xi,0) &= g_0^{(2)}(\xi), \qquad u(\xi,1) = g_1^{(2)}(\xi), \end{cases}$$

where $a(\xi,\eta)$ is a positive and piecewise differentiable function, $\theta(\xi,\eta)$ is a nonnegative bounded function, and $g_0^{(1)}(\eta)$, $g_1^{(1)}(\eta)$, $g_0^{(2)}(\xi)$, $g_1^{(2)}(\xi)$ and $f(\xi,\eta)$ are bounded functions. The case that $a(\xi,\eta)=1$, $\theta(\xi,\eta)=0$ and $g_0^{(1)}(\eta)=g_1^{(1)}(\eta)=g_0^{(2)}(\xi)=g_1^{(2)}(\xi)=0$ has been extensively studied in literatures, e.g., [1, 12, 15, 16]. In this paper, we will study the case that

$$g_0^{(1)}(\eta) = g_1^{(1)}(\eta) \equiv g^{(1)}(\eta), \tag{1.2}$$

i.e., the boundary conditions are periodic on the ξ-direction and Dirichlet on the η-direction, respectively. Moreover, for simplicity but without loss of generality, we assume that $\theta(\xi,\eta)=0$ and $g_0^{(2)}(\xi)=g_1^{(2)}(\xi)\equiv 0$ in the sequel.

* Received November 1, 2002; final revised April 20, 2004.
1) Subsidized by The Special Funds For Major State Basic Research Projects (No. G1999032803) and the National Natural Science Foundation (No. 10471146).

When the second-order self-adjoint elliptic Dirichlet-periodic boundary value problem (1.1)-(1.2) is discretized by the five-point central difference scheme with mesh size $h = \frac{1}{N+1}$, associated with the interior mesh point (ih, jh) we have the difference equation

$$s_{i,j}u_{i,j} - a_{i-\frac{1}{2},j}u_{i-1,j} - a_{i+\frac{1}{2},j}u_{i+1,j} - a_{i,j-\frac{1}{2}}u_{i,j-1} - a_{i,j+\frac{1}{2}}u_{i,j+1} = h^2 f_{i,j},$$

where

$$s_{i,j} = a_{i-\frac{1}{2},j} + a_{i+\frac{1}{2},j} + a_{i,j-\frac{1}{2}} + a_{i,j+\frac{1}{2}},$$

and for $j = 1, 2, \ldots, N$, we stipulate that $a_{(N+i)+\frac{1}{2},j} = a_{i-\frac{1}{2},j}$ in the light of the periodicity of the boundary condition (1.2). By arranging the unknowns $\{u_{i,j}\}_{1\leq i\leq N+1, 1\leq j\leq N}$ according to the natural ordering and letting $n = (N+1)N$, we obtain the system of linear equations:

$$\mathbf{A}x = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{n\times n} \text{ symmetric positive definite, and } \mathbf{b} \in \mathbb{R}^n, \tag{1.3}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & B_1 & & & \\ B_1 & \mathbf{A}_2 & B_2 & & \\ & \ddots & \ddots & \ddots & \\ & & B_{N-2} & \mathbf{A}_{N-1} & B_{N-1} \\ & & & B_{N-1} & \mathbf{A}_N \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} h^2 f_{1,1} \\ h^2 f_{1,2} \\ \vdots \\ h^2 f_{N+1,N-1} \\ h^2 f_{N+1,N} \end{pmatrix}, \tag{1.4}$$

and for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, N-1$,

$$\mathbf{A}_i = \begin{pmatrix} a_1^{(i)} & d_1^{(i)} & & & \sigma^{(i)} \\ d_1^{(i)} & a_2^{(i)} & d_2^{(i)} & & \\ & \ddots & \ddots & \ddots & \\ & & d_{N-1}^{(i)} & a_N^{(i)} & d_N^{(i)} \\ \sigma^{(i)} & & & d_N^{(i)} & a_{N+1}^{(i)} \end{pmatrix}, \quad B_j = \begin{pmatrix} b_1^{(j)} & & & & \\ & b_2^{(j)} & & & \\ & & \ddots & & \\ & & & b_N^{(j)} & \\ & & & & b_{N+1}^{(j)} \end{pmatrix}. \tag{1.5}$$

The sub-matrices $\mathbf{A}_i \in \mathbb{R}^{(N+1)\times(N+1)} (i = 1, 2, \ldots, N)$ are symmetric positive definite whose elements are defined by

$$a_j^{(i)} = s_{j,i}, \quad d_j^{(i)} = -a_{j+\frac{1}{2},i}, \quad \sigma^{(i)} = -a_{i-\frac{1}{2},i};$$

and the sub-matrices $B_i \in \mathbb{R}^{(N+1)\times(N+1)} (i = 1, 2, \ldots, N-1)$ are diagonal whose elements are defined by

$$b_j^{(i)} = -a_{j,i+\frac{1}{2}}.$$

Clearly, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is an irreducibly diagonally dominant Z-matrix. Therefore, it is an M-matrix. And so are the sub-matrices $\mathbf{A}_i$ $(i = 1, 2, \ldots, N)$. We refer the readers to [17, 18] for details.

The preconditioned conjugate gradient (PCG) method[11, 7, 10] is one of the most powerful methods for getting an accurate approximation to the solution $x^* \in \mathbb{R}^n$ of the system of linear equations (1.3). As a matter of fact, if a symmetric positive definite matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ is employed as a preconditioner to the coefficient matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, then the corresponding PCG iteration converges to x^* within a relative error ε in at most $\frac{1}{2}\sqrt{\kappa(\mathbf{M}^{-1}\mathbf{A})}\ln\frac{2}{\varepsilon} + 1$ number of iteration steps[2], where $\kappa(\mathbf{M}^{-1}\mathbf{A})$ represents the Euclidean condition number of the preconditioned matrix $\mathbf{M}^{-1}\mathbf{A}$. See also [9, 10, 4, 6]. Therefore, a good preconditioner is the key factor to considerably improve the convergence behaviour of the PCG iteration.

As we know, standard preconditioners to a symmetric positive definite matrix may be constructed by the *incomplete Cholesky* (IC) factorization[2, 10] and the *symmetric successive overrelaxation* (SSOR) iteration[17, 18, 1] techniques. See also [3, 5, 8, 15, 16]. However, these two classes of preconditioners are only applicable and efficient for a special class of symmetric