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THE DERIVATIVE ULTRACONVERGENCE FOR QUADRATIC TRIANGULAR FINITE ELEMENTS [*1)]

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Abstract

This work concerns the ultraconvergence of quadratic finite element approximations of elliptic boundary value problems. A new, discrete least-squares patch recovery technique is proposed to post-process the solution derivatives. Such recovered derivatives are shown to possess ultraconvergence. The keys in the proof are the asymptotic expansion of the bilinear form for the interpolation error and a "localized" symmetry argument. Numerical results are presented to confirm the analysis.

Mathematics subject classification: 65N30.
Key words: Ultra-closeness, Superconvergence patch recovery (SPR), Ultraconvergence.

1. Introduction

The superconvergence of the derivatives for quadratic triangular finite element has been studied in the pointwise sense since 1981. And based on the theory of the discrete Green's function and two basic estimates[15]~[20], a usual superconvergence order, $O(h^3)$, has been obtained. However, the *superconvergence patch recovery (SPR)* introduced by *Zienkiewics and Zhu*[21]~[23] indicates that, by using the least-squares and the interpolation process, the recovered nodal values of the derivatives are superconvergent, in particular those for quadratic elements are ultraconvergent, i.e., with order of $O(h^4)$. The above results for the two-point boundary value problem and rectangular elements have been discussed a great deal in [9] ~ [14], but the ultraconvergence for quadratic triangular finite elements is so challenging that nobody has proved it up to now.

Our paper obtained the ultra-closeness results by the asymptotic expansion proposed in [2], and proceeded to verify the important problem.

For simplicity, we consider the model problem: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,\, v),\ \forall\, v \in H_0^1(\Omega), \tag{1.1}$$

where $a(u,v) = (\nabla u, \nabla v)$, $(f,v) = \displaystyle\int_\Omega f \cdot v dxdy$, Ω is a smooth or convex polygonal domain. It is easy to see from the PDE theory that there exists $2 < q_0 < \infty$ such that the mapping

$$\Delta :\quad W^{2,\,q}(\Omega) \cap W_0^{1,p}(\Omega) \to L^q(\Omega) \tag{1.2}$$

is a homeomorphic one for any $q \in (1, q_0)$.

Let $\mathcal{T}^h$ denote a uniform triangulation of Ω (only a local uniform triangulation is needed while considering the local superconvergence in this paper), $S_0^h(\Omega)$ a finite element space over $\mathcal{T}^h$ consisting of piecewise polynomials of degree 2. For each triangular element $e \in \mathcal{T}^h$, it suffices to let $e = \Delta z_1 z_2 z_3$, and define by $P_k(e)$ the set of the polynomials of degree k on e, and define $P_k^0(e) = \{p \in P_k(e) : p|_{\partial e} = 0\}$. Furthermore, to distinguish we add "′" to the

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corresponding term depending on e'. For a point z, $U_d(z)$ denotes the neighborhood of the point z, whose center is z and whose radius is d.

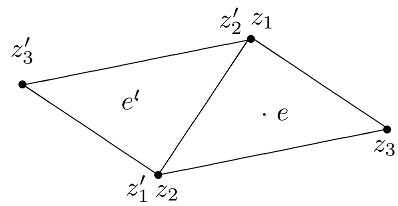

Fig. 1

Point z is said to be locally symmetric for the subdivision $\mathcal{T}^h$ if there exists a neighborhood, $U_d(z)$, of z such that the mesh covering this neighborhood is symmetric with respect to the point z. Generally, denote by E_z the union of the elements surrounding the point z.

It is clear that the vertexes of each element and the middle point of each side are the locally symmetric points.

Let ∂_z denote the direction derivative operator in the oriented direction l, and

$$\bar{\partial}_z v(z) = \frac{1}{2}[\lim_{t\to+0}\partial_z v(z+tl) + \lim_{t\to+0}\partial_z v(z-tl)],$$

is said to be the average derivative in the direction l for v. In the meantime, we may similarly define the gradient average operator $\bar{\nabla}$. And $\int_\Omega' f dxdy = \sum_{e\in\mathcal{T}^h}\int_e f dxdy$, then $\|f\|_m' = (\sum_{e\in\mathcal{T}^h}\|f\|_{m,\,e}^2)^{\frac{1}{2}}$.

Throughout this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for *Sobolev* space on $\Omega \subset R^2$ with norm $\|\cdot\|_{W^{m,p}(\Omega)}$. Particularly, we denote $W^{m,2} = H^m$. In addition, C denotes a nonnegative constant independent of h, u unless additional explanation, which can have different value in different place.

It is well known that all superconvergent estimates are in relation to pollution of boundary condition, but these pollution can be controlled by the negative norm

$$\|u-u^h\|_{-s,\,\Omega} = \sup_{v\in H_0^s(\Omega)}\frac{(u-u^h,\,v)}{\|v\|_s}, s>0,$$

that is

Proposition 1. *Suppose that* $w \in S^h(\Omega)$ *satisfy*

$$a(w,v)=0,\ \forall v\in S_0^h(U_d(z)),$$

then

$$|\bar{\nabla}w(z)| \le C\|w\|_{-s,U_d(z)}.$$

Strang[6], *Nitsche* and *Schatz*[3] have achieved the following negative norm estimates:

$$\|u-u^h\|_{-s,\Omega} \le Ch^{2k}\|u\|_{k+1,\,\Omega},$$

where k is the degree of finite element, in particular, $k=2$ in this paper. This shows the pollution does rarely affect interior and local superconvergence, and only local ultraconvergence is considered in this paper, so it is sufficient to consider the case that Ω be rectangular domain.

We have proved the following in the monograph [2] (cf. Theorem 3.5.1):

Proposition 2. *Let* $\mathcal{T}^h$ *be a uniform triangulation of* Ω*,* $S_0^h(\Omega)$ *a quadratic finite element space over* $\mathcal{T}^h$*, then*

$$\begin{aligned} a(u-u^I,v) = \quad & h^4\int_\Omega'(C\cdot D^4u\cdot D^2v + C\cdot D^5u\cdot Dv)dxdy \\ & +O(h^5)\|u\|_{5,\infty}\|v\|_{2,1}',\ \forall v\in S_0^h(\Omega), \end{aligned} \tag{1.3}$$