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THE NEAREST BISYMMETRIC SOLUTIONS OF LINEAR MATRIX EQUATIONS *1)

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Abstract

The necessary and sufficient conditions for the existence of and the expressions for the bisymmetric solutions of the matrix equations (I) $A_1X_1B_1+A_2X_2B_2+\cdots+A_kX_kB_k=D$, (II) $A_1XB_1+A_2XB_2+\cdots+A_kXB_k=D$ and (III) $(A_1XB_1, A_2XB_2, \cdots, A_kXB_k) = (D_1, D_2, \cdots, D_k)$ are derived by using Kronecker product and Moore-Penrose generalized inverse of matrices. In addition, in corresponding solution set of the matrix equations, the explicit expression of the nearest matrix to a given matrix in the Frobenius norm is given. Numerical methods and numerical experiments of finding the nearest solutions are also provided.

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1. Introduction

Denote by R^n the set of all real n-component vectors, $R^{m\times n}$ the set of all $m\times n$ real matrices and $BSR^{n\times n}$ the set of all $n\times n$ real bisymmetric matrices (A symmetric matrix $A=(a_{ij})\in R^{n\times n}$ is called bisymmetric if $a_{ij}=a_{n+1-j,n+1-i}$ for all $1\le i,j\le n$). I_n represents the $n\times n$ identity matrix. $\|A\|_F, A^+$ and A^T stand for the Frobenius norm, Moore-Penrose generalized inverse and transpose of a matrix A, respectively. On $R^{m\times n}$ we define inner product: $\langle A,B\rangle = trace(B^TA)$ for all $A,B\in R^{m\times n}$, then $R^{m\times n}$ is a Hilbert inner product space and the norm of a matrix generated by this inner product is Frobenius norm. For $A=(a_{ij})\in R^{m\times n}, B=(b_{ij})\in R^{p\times q}$, let $A\bigotimes B\in R^{mp\times nq}$ be the Kronecker product of A and B.

Various aspects for the solution of linear matrix equations have been investigated. For example, Baksalary and Kala [1], Chu [4], He [8], and Xu, Wei and Zheng [13] considered the nonsymmetric solution of the matrix equation $AXB+CXD=E$ by using Moore-Penrose generalized inverse and the generalized singular value decomposition of matrices, while Chang and Wang [3], Jameson [9] and Dai [6] considered the symmetric conditions on the solution of the matrix equations: $AXA^T+BYB^T=C$, $AX+YA=C$, $AX=YB$ and $AXB=C$. Zietak [14, 15] discussed the l_p-solution and chebyshev-solution of the matrix equation $AX+YB=C$. Dobovisek [7] discussed the minimal solution of the matric equation $AX-YB=0$. Chu [5], and Kucera [11] and Jameson [10] are, respectively, studied the nonsymmetric solution of the matrix equation $AXB+CXD=E$ and its special case $AX+XB=C$. Mitra [12], Chu [4] and the references therein studied the nonsymmetric solution of the matrix equation $(AXB, CXD)=(E,F)$.

In this paper, the following problems are considered

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Problem I. Given $X_i^* \in R^{n_i \times n_i}, A_i \in R^{p \times n_i}, B_i \in R^{n_i \times q}$ $(i = 1, 2, \cdots, k)$ and $D \in R^{p \times q}$. Let

$$H_1 = \{[X_1, X_2, \cdots, X_k] : A_1X_1B_1 + A_2X_2B_2 + \cdots + A_kX_kB_k = D, X_i \in BSR^{n_i \times n_i}\}, \quad (1.1)$$

find $[\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_k] \in H_1$ such that

$$\begin{aligned} \|[\hat{X}_1, \cdots, \hat{X}_k] - [X_1^*, \cdots, X_k^*]\|_F \equiv & \ (\|\hat{X}_1 - X_1^*\|_F^2 + \|\hat{X}_2 - X_2^*\|_F^2 + \cdots + \|\hat{X}_k - X_k^*\|_F^2)^{\frac{1}{2}} \\ = & \ \min_{[X_1, \cdots, X_k] \in H_1} \|[X_1, \cdots, X_k] - [X_1^*, \cdots, X_k^*]\|_F. \quad (1.2) \end{aligned}$$

Problem II. Given $X^* \in R^{n \times n}, A_i \in R^{p \times n}, B_i \in R^{n \times q}$ $(i = 1, 2, \cdots, k)$ and $D \in R^{p \times q}$. Let

$$H_2 = \{X \in BSR^{n \times n} : A_1XB_1 + A_2XB_2 + \cdots + A_kXB_k = D\}, \quad (1.3)$$

find $\hat{X} \in H_2$ such that

$$\|\hat{X} - X^*\|_F = \min_{X \in H_2} \|X - X^*\|_F. \quad (1.4)$$

Problem III. Given $X^* \in R^{n \times n}, A_i \in R^{p_i \times n}, B_i \in R^{n \times q_i}$ and $D_i \in R^{p_i \times q_i}$ $(i = 1, 2, \cdots, k)$. Let

$$H_3 = \{X \in BSR^{n \times n} : A_1XB_1 = D_1, A_2XB_2 = D_2, \cdots, A_kXB_k = D_k\}, \quad (1.5)$$

find $\hat{X} \in H_3$ such that

$$\|\hat{X} - X^*\|_F = \min_{X \in H_3} \|X - X^*\|_F. \quad (1.6)$$

Using Kronecker product and Moore-Penrose generalized inverse of matrices, the necessary and sufficient conditions for the existence of and the explicit expressions for the solution of Problem I, II and III are derived. Numerical methods and numerical experiments of finding the nearest solutions are also provided.

2. Solving Problems I, II and III

For matrix $A \in R^{m \times n}$, denotes by $vec(A)$ the following vector containing all the entries of matrix A:

$$vec(A) = [A(1,:), A(2,:), \cdots, A(n,:)]^T \in R^{mn}, \quad (2.1)$$

where $A(i,:)$ denote ith row of matrix A. For vector $\mathbf{x} \in R^{n^2}$, denote by $vec_n^{-1}(\mathbf{x})$ the following matrix containing all the entries of vector $\mathbf{x}$:

$$vec_n^{-1}(\mathbf{x}) = \begin{pmatrix} \mathbf{x}(1:n)^T \\ \mathbf{x}(n+1:2n)^T \\ \vdots \\ \mathbf{x}[(n-1)n+1:n^2]^T \end{pmatrix} \in R^{n \times n}, \quad (2.2)$$

where $\mathbf{x}(i:j)$ denotes elements i to j of vector $\mathbf{x}$.

Let

$$vec(BSR^{n \times n}) = \{vec(A) : A \in BSR^{n \times n}\} \subset R^{n^2}, \quad (2.3)$$

then the dimension of the subspace $vec(BSR^{n \times n})$ is $r = (n+1)^2/4$ when n is add or $r = n(n+2)/4$ when n is even. Suppose $w_1, w_2, \cdots, w_r$ is an orthonormal basis-set for $vec(BSR^{n \times n})$. For example, suitable w_i for $n = 3$ might be $w_1 = [\frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, 0, \frac{1}{\sqrt{2}}]^T$, $w_2 = [0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0]^T$, $w_3 = [0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0]^T$, $W_4 = [0, 0, 0, 0, 1, 0, 0, 0, 0]^T$. Consequently

$$W = [w_1, w_2, \cdots, w_r] \in R^{n^2 \times r} \quad (2.4)$$