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CASCADIC MULTIGRID FOR FINITE VOLUME METHODS FOR ELLIPTIC PROBLEMS *1)

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Abstract

In this paper, some effective cascadic multigrid methods are proposed for solving the large scale symmetric or nonsymmetric algebraic systems arising from the finite volume methods for second order elliptic problems. It is shown that these algorithms are optimal in both accuracy and computational complexity. Numerical experiments are reported to support our theory.

Mathematics subject classification: 65N55.
Key words: Cascadic multigrid, Finite volume methods, Elliptic problems.

1. Introduction

The finite volume methods or covolume methods have become powerful tools for numerically solving PDEs. They can also be termed as box methods [1], generalized finite difference methods [15]. These methods have a simplicity for implementation comparable to the finite difference methods; on the other hand, they have a flexibility similar to that of finite element methods for handling complicated geometries and boundary conditions. Another important advantage of these methods is that the numerical solutions usually have certain conservation property, which are very desirable in many applications, especially in CFD. For a comprehensive presentation and more references of existing results in this direction, we refer to the monographs [15],[13], for details.

The algebraic systems resulting from the finite volume methods are sparse and ill-conditioned. So we should construct some effective methods like multigrid methods or domain decomposition methods for solving such kind of large scale systems. Although the convergence behavior of multigrid algorithms for standard finite element methods is by now well understood, much less is known for the finite volume element method. Recently, a V-cycle multigrid for the finite volume element method was proposed in [10] by Chou and Kwak. They show that the multigrid is optimal, which means that the convergence rate of this method is independent of the mesh size and mesh level. The aim of this paper is to present some cascadic multigrid algorithms for the discretization systems of the finite volume methods. Compared with usual multigrid, the main advantage of the cascadic multigrid method is its simplicity[2][17]. It requires no coarse grid corrections at all and may be viewed as a "one-way" multigrid method. In recent years, there have been several theoretical analysis and the applications of these methods, cf. [17][19]for nonconforming element methods and plate bending problems, [18] for parabolic problems, [14][20] for nonlinear problems, [5] for Stokes problems, [6] for mortar element methods.

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In this paper, we shall first propose a cascadic multigrid algorithm for the symmetric systems resulting from finite volume method approximation of some special second order elliptic equations. In this case, the quadratic forms in different mesh levels are noninherited. We shall show that the cascadic multigrid algorithm holds optimal accuracy and computational complexity. Second, it is known that the algebraic equations arising from the finite volume methods are usually nonsymmetric, which brings us many difficulties for designing an optimal cascadic multigird algorithms. But note that the nonsymmetric equations are a small perturbation of the usual finite element discretization equations. Based on this observation, we shall construct an efficient cascadic multigrid algorithm for this large scale nonsymmetric system. In this algorithm, we shall first solve a small nonsymmetric problem on the coarsest grids which is associated with low frequencies of the discretization system, and then solve symmetric positive definite (SPD) finite element problems on the fine levels. Under this construction, we shall also prove that the cascadic multigrid is optimal in both the accuracy and computational complexity.

The rest of our paper is organized as follows: In Section 2, we give some notations used in this paper and formulate the finite volume element schemes. In Section 3, the cascadic multigrid methods for the symmetric and nonsymmetric systems are analyzed respectively. In the last section, numerical experiments are reported to support our theory.

2. A Model Problem and Finite Volume Methods

We consider the following self-adjoint elliptic problem

$$\begin{cases} -\nabla\cdot(A\nabla u)+qu &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a polygonal domain; $f \in L^2(\Omega)$, $q \in L^\infty(\Omega)$ and $q \geq 0$ almost everywhere in Ω are two given real-valued functions; $A = (a_{ij})_{2\times 2} \in (W^{1,\infty}(\Omega))^4$ is a given real symmetric matrix-valued function. We assume that A satisfies the following ellipticity condition: there exists a constant $\alpha_1 > 0$ such that

$$\alpha_1 \xi^T\xi \leq \xi^T A(\mathbf{x})\xi, \qquad \forall\, \xi \in R^2 \text{ and } \mathbf{x} = (x, y) \in \bar{\Omega}. \tag{2.2}$$

In what follows we shall adopt the standard definitions of Sobolev spaces and their norms and semi-norms as presented in [11].

The variational formulation associated with (2.1) is to find $u \in V = H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in V, \tag{2.3}$$

where

$$\begin{aligned} a(u, v) &= \int_\Omega (A\nabla u\cdot\nabla v + quv)dx, \\ (f, v) &= \int_\Omega fv dx. \end{aligned}$$

Under the above assumptions, it is known that (2.3) holds a unique solution $u \in H^2(\Omega)\cap H_0^1(\Omega)$.

Define the energy norm as:

$$\|v\|_a = a(v, v)^{\frac{1}{2}}, \quad \forall v \in H_0^1(\Omega).$$

It is easy to check that this norm is equivalent to the usual norm $\|\cdot\|_1$ over the space $H_0^1(\Omega)$.

In order to present the cascadic multigrid algorithm for finite volume methods, we first construct a sequence of nested triangulations of Ω as follows. Suppose that a coarse triangulation $\mathcal{T}_0$ of Ω is given, we define the finer triangulation $\mathcal{T}_l$ for $l \geq 1$ by subdividing a triangle in $\mathcal{T}_{l-1}$ into four subtriangles by connecting the midpoints of the edges. Assume that the coarse