Journal of Computational Mathematics, Vol.23, No.1, 2005, 27–36.

SUPERCONVERGENCE OF TETRAHEDRAL QUADRATIC FINITE ELEMENTS *[1)]

Jan Brandts
(*Korteweg-de Vries Institute for Mathematics, Faculty of Science, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, Netherlands*)

Michal Křížek
(*Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic*)

Abstract

For a model elliptic boundary value problem we will prove that on strongly regular families of uniform tetrahedral partitions of a pohyhedral domain, the gradient of the quadratic finite element approximation is superclose to the gradient of the quadratic Lagrange interpolant of the exact solution. This supercloseness will be used to construct a post-processing that increases the order of approximation to the gradient in the global L^2-norm.

Mathematics subject classification: 65N30.
Key words: Tetrahedron, Superconvergence, Supercloseness, Post-processing, Gauss points.

1. Introduction

The topic of this paper is supercloseness and superconvergence of a finite element that is frequently used in practical applications: the tetrahedral quadratic element. As a matter of fact, in the engineering society, it is even more popular than the linear element, in spite of the fact that the latter has been studied in much more detail. Before we discuss our results in Section 1.2, we will introduce the term superconvergence and put it in its historical context in Section 1.1. In particular, we comment on work by other authors having direct links to our results.

1.1. Overview

Superconvergence of standard and mixed finite elements is a well-known and practically useful topic in finite element analysis. Usually, a finite element method is called superconvergent, if at special points (or on special lines) the rate of convergence is higher than what is globally possible (cf. [11, 13, 23, 27, 28]). Oganesjan and Ruhovets [24] proved that for linear triangular elements on uniform partitions the gradient of the finite element approximation is a higher order perturbation of the gradient of a local interpolant of the exact solution. This property, which lies also at the basis of the papers (cf. [4, 9, 10, 12, 19, 22, 29]), is often called supercloseness. In both cases, one can usually construct, without too much additional computational effort, approximations that are globally better than the original one. This procedure is called post-processing. The difference between the original and the post-processed approximation may then be used as an asymptotically exact error estimation. For some interesting papers and an abundance of references, we refer to [21].

* Received October 10, 2002; final revised June 4, 2003.
[1)] Jan Brandts was supported by a Fellowship of the Royal Netherlands Academy of Arts and Sciences. Michal Křížek was supported by Grant No. 201/02/1058 of the Grant Agency of the Academy of Sciences of the Czech Republic.

Already in 1981, Zhu [30] proved superconvergence of the gradient of quadratic triangular elements on uniform triangulations, so, in the two-dimensional case. In [31], he discusses superconvergence at nodal points for this setting. Later, similar results were obtained by Andreev in [1] and Andreev and Lazarov in [2], who proved that the tangential component of the gradient is superconvergent at the two Gauss points at each edge of each triangle (see also Goodsell and Whiteman [16, 17]). Recently, Brandts rederived some of these results in [5] as a by-product of a superconvergence proof for one-but-lowest order Raviart-Thomas mixed finite elements.

Superconvergence results for three-dimensional problems are relatively scarce, since not all techniques for the two-dimensional case can be generalized. Typical difficulties with superconvergence in $\mathbb{R}^3$ are surveyed in [6]. As far as we know, the Chinese were the first to prove superconvergence for the gradient in a three-dimensional setting. In 1980, Chen considered linear elements on tetrahedra in [7], which was followed in the second half of the eighties by Kantchev and Lazarov with the paper [18]. A short note by Pehlivanov [25] reflects on the quadratic case, but unfortunately without any (reference to a) proof. In 1994, Goodsell derived, for the gradient, pointwise superconvergence results for linear tetrahedral elements in [15]. Finally, results in [26, 27] imply superconvergence at nodal values for quadratic three-dimensional elements on locally point-symmetric meshes. In fact, we will need a result from [26] in our proofs. In the two-dimensional case [5], this was not necessary because of favorable properties of a mixed finite element Fortin interpolation, which do not generalize to the three-dimensional case (see [6], p. 29).

1.2. Outline

The Poisson equation with homogeneous Dirichlet boundary conditions will be our model elliptic problem. We employ regular family of uniform tetrahedral partitions of the domain. The gradient of the standard quadratic finite element approximations will then be proved superclose to the gradient of the nodal quadratic Lagrange interpolant of the exact solution. Once more we stress that the Fortin-like interpolant that was used in the two-dimensional setting, has no special advantages in 3D as in 2D. See [5] and [6] for details.

The outline of this paper is as follows. Section 2 contains some preliminaries. In Section 3, we derive auxiliary results for so-called quadratic bubble functions, which will be used in the proofs of our main results in Section 4. There we prove the supercloseness between the gradients of the finite element approximation and the quadratic Lagrange interpolant of the exact solution. A numerical test is presented for illustration. In Section 5, we discuss the generalization of the results to other elliptic problems with varying coefficients, and the extension of a post-processing scheme by Andreev and Lazarov [2], which will lead to a higher order approximation of the gradient.

2. Preliminaries

Let Ω be a bounded polyhedral domain in $\mathbb{R}^3$ with Lipschitz boundary. Denote by $H^k(\Omega)$ the usual Sobolev spaces of functions having generalized partial derivatives up to order k in $L^2(\Omega)$ and their usual norm and seminorm by $\|\cdot\|_k$ and $|\cdot|_k$, respectively. The subspace of functions from $H^1(\Omega)$ with vanishing traces on $\partial\Omega$ we denote by $H_0^1(\Omega)$. Before turning to the discrete spaces, we will elaborate on uniform partitions of the domain.

2.1. Uniform partitions of a domain into tetrahedra

A triangulation of a planar domain is called *uniform* if the union of any two triangles sharing an entire edge forms a parallelogram. The feature of interest is, that a parallelogram is a set that is invariant under reflection in its center of gravity. For brevity, we will refer to such sets as "point-symmetric sets". Since two tetrahedra having a face in common never form a