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STABILITY OF GENERAL LINEAR METHODS FOR SYSTEMS OF FUNCTIONAL-DIFFERENTIAL AND FUNCTIONAL EQUATIONS *1)

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Abstract

This paper is concerned with the numerical solution of functional-differential and functional equations which include functional-differential equations of neutral type as special cases. The adaptation of general linear methods is considered. It is proved that A-stable general linear methods can inherit the asymptotic stability of underlying linear systems. Some general results of numerical stability are also given.

Mathematics subject classification: 65L05.
Key words: Hybrid systems, Functional-differential equations, Functional equations, General linear methods, Numerical stability.

1. Introduction

Neutral functional differential equations with one state-independent time delay are usually formulated in the form

$$y'(t) = f(t, y(t), y(\phi(t)), y'(\phi(t))), \quad t \geq 0, \tag{1.1}$$

where f and ϕ are given functions with $\phi(t) \leq t$ for $t \geq 0$.

In contrast to (1.1) there are neutral functional differential equations of the form

$$[z(t) - g(t, z(\phi(t)))]' = f(t, z(t), z(\phi(t))), \quad t \geq 0, \tag{1.2}$$

where f, g and ϕ are given functions with $\phi(t) \leq t$ for $t \geq 0$. To distinguish (1.2) from (1.1), Liu [17] calls (1.1) an explicit neutral equation and (1.2) an implicit neutral equation. It is obvious that (1.1) and (1.2) are equivalent to

$$\begin{cases} y'(t) = f(t, y(t), y(\phi(t)), z(\phi(t))), \\ z(t) = f(t, y(t), y(\phi(t)), z(\phi(t))), \end{cases} \quad t \geq 0 \tag{1.3}$$

and

$$\begin{cases} y'(t) = f(t, y(t) + g(t, z(\phi(t))), z(\phi(t))), \\ z(t) = y(t) + g(t, z(\phi(t))), \end{cases} \quad t \geq 0, \tag{1.4}$$

respectively.

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The numerical solution for neutral functional-differential equations (1.1) has been studied extensively in recent years (cf. [1, 7, 12, 13, 19, 22]). There seems to be little difference between solving (1.1) numerically and solving (1.3) numerically. However, the situation with the implicit neutral equation (1.2) is very different. Even in the case where (1.2) can be transformed into the form of (1.1), there are certain advantages in solving it by formulating numerical schemes based on its equivalent form (1.4). This has been discussed by Liu [17].

More general form than (1.3) and (1.4) is

$$\begin{cases} y'(t) = f(t, y(t), y(\phi(t)), z(\phi(t))), \\ z(t) = g(t, y(t), y(\phi(t)), z(\phi(t))), \end{cases} t \geq 0. \tag{1.5}$$

It is easily seen that

$$\begin{cases} y'(t) = f(t, y(t), z(t), y(\phi(t)), z(\phi(t))), \\ z(t) = g(t, y(t), y(\phi(t)), z(\phi(t))), \end{cases} t \geq 0 \tag{1.6}$$

can be transformed into the form of (1.5). Systems of the form (1.5) are sometimes called hybrid systems [11] or systems of functional-differential and functional equations [18]. The form of (1.5) includes functional-differential equations of neutral type (1.1) and (1.2) as special cases.

In order to investigate the linear stability of numerical methods to (1.5), Liu [18] considers the following test problems

$$\begin{cases} y'(t) + A_1 y(t) + A_2 y(t-\tau) + B_1 z(t-\tau) = 0, \\ z(t) + A_3 y(t) + A_4 y(t-\tau) + B_2 z(t-\tau) = 0, \end{cases} t \geq 0, \tag{1.7a}$$

with the initial conditions

$$y(t) = \varphi(t), z(t) = \psi(t), \quad t \leq 0, \tag{1.7b}$$

where $\tau > 0$, and $A_1, A_2 \in C^{d_1 \times d_1}, A_3, A_4 \in C^{d_2 \times d_1}, B_1 \in C^{d_1 \times d_2}, B_2 \in C^{d_2 \times d_2}$ are the coefficient matrices, φ, ψ are given vectors of complex functions that satisfy the consistency condition

$$\psi(0) + A_3\varphi(0) + A_4\varphi(-\tau) + B_2\psi(-\tau) = 0.$$

We introduce some notations. $\sigma(A), \rho(A)$ and $\alpha(A)$ for a matrix A designate the spectrum, spectral radius and maximal real parts of the eigenvalues of A, respectively,

$$P(z,\xi) = \begin{vmatrix} zI_{d_1} + A_1 + A_2\xi & B_1\xi \\ A_3 + A_4\xi & I_{d_2} + B_2\xi \end{vmatrix}.$$

Other notation include

$$C^+ = \{z \in C \big| \mathrm{Re} z > 0\}, C^0 = \{z \in C \big| \mathrm{Re} z = 0\}, C^- = \{z \in C \big| \mathrm{Re} z < 0\},$$
$$D = \{z \in C \big| |z| < 1\}, \quad \Gamma = \{z \in C \big| |z| = 1\}.$$

In [18], it is shown that the initial-value problem (1.7) is asymptotically stable for every $\tau > 0$ if and only if

$$\rho(B_2) < 1, \tag{1.8a}$$

$$P(z,\xi) \neq 0 \text{ for all } z \in C^0 \setminus \{0\} \text{ and } \xi \in \Gamma, \tag{1.8b}$$

$$\alpha(B_1(I_{d_2} + B_2)^{-1}(A_3 + A_4) - A_1 - A_2) < 0. \tag{1.8c}$$

Moreover, the asymptotical stability of (1.7) implies the following are true [18]:

(1)$\alpha(-A_1) < 0$,

(2)$P(z,\xi) \neq 0$ for all $z \in C^0$ and $\xi \in D$,