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WAVEFORM RELAXATION METHODS OF NONLINEAR INTEGRAL-DIFFERENTIAL-ALGEBRAIC EQUATIONS *1)

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Abstract

In this paper we consider continuous-time and discrete-time waveform relaxation methods for general nonlinear integral-differential-algebraic equations. For the continuous-time case we derive the convergence condition of the iterative methods by invoking the spectral theory on the resulting iterative operators. By use of the implicit difference forms, namely the backward-differentiation formulae, we also yield the convergence condition of the discrete waveforms. Numerical experiments are provided to illustrate the theoretical work reported here.

Mathematics subject classification: 37M05, 45J05, 65L80, 65Y05.
Key words: Nonlinear integral-differential-algebraic equations, Waveform relaxation, Parallel solutions, Convergence of iterative methods, Engineering applications.

1. Introduction

We consider a system which is described by nonlinear integral-differential-algebraic equations (IDAEs) as follows

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \tilde{f}_1(\dot{x}(t), x(t), y(t), \int_0^t \tilde{h}_1(x(s), y(s), s, t)ds, t), \quad x(0) = x_0, \\ y(t) & = & \tilde{f}_2(\dot{x}(t), x(t), y(t), \int_0^t \tilde{h}_2(x(s), y(s), s, t)ds, t), \quad t \in [0, T], \end{array}\right. \tag{1}$$

where t is the time variable, $x_0 \in \mathbf{R}^n$ is an initial value, $[0, T]$ is a given finite time interval, $x(t) \in \mathbf{R}^n$ and $y(t) \in \mathbf{R}^m$ are to be computed. We assume that the initial condition of (1) is consistent, that is, for a given $x_0(= x(0))$ we can solve out $\dot{x}(0)$ and $y(0)$ from the following initial condition system:

$$\left\{\begin{array}{rcl} \dot{x}(0) & = & \tilde{f}_1(\dot{x}(0), x_0, y(0), 0, 0), \\ y(0) & = & \tilde{f}_2(\dot{x}(0), x_0, y(0), 0, 0). \end{array}\right. \tag{2}$$

We will also denote $y(0)$ as y_0 in this paper.

For a large and complex system like (1) waveform relaxation (WR) or dynamic iteration is a novel parallel algorithm of treating its numerical solutions [1 - 6]. Numerical algorithms with WR are suitable to be processed in parallel [7]. The general form of the WR algorithm for (1) is

$$\left\{\begin{array}{rcl} \dot{x}^{(k+1)}(t) & = & f_1(\dot{x}^{(k+1)}(t), \dot{x}^{(k)}(t), x^{(k+1)}(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), \\ & & \int_0^t h_1(x^{(k+1)}(s), x^{(k)}(s), y^{(k+1)}(s), y^{(k)}(s), s, t)ds, t), \\ y^{(k+1)}(t) & = & f_2(\dot{x}^{(k+1)}(t), \dot{x}^{(k)}(t), x^{(k+1)}(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), \\ & & \int_0^t h_2(x^{(k+1)}(s), x^{(k)}(s), y^{(k+1)}(s), y^{(k)}(s), s, t)ds, t), \\ & & x^{(k+1)}(0) = x_0, \quad t \in [0, T], \quad k = 0, 1, \cdots, \end{array}\right. \tag{3}$$

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where $[x^{(0)}(\cdot), y^{(0)}(\cdot)]^t$ is an initial guess, the nonlinear splitting functions $f_1 : (\mathbf{R}^n)^4 \times (\mathbf{R}^m)^2 \times \mathbf{R}^{l_1} \times [0,T] \to \mathbf{R}^n$, $f_2 : (\mathbf{R}^n)^4 \times (\mathbf{R}^m)^2 \times \mathbf{R}^{l_2} \times [0,T] \to \mathbf{R}^m$, $h_1 : (\mathbf{R}^n)^2 \times (\mathbf{R}^m)^2 \times [0,T]^2 \to \mathbf{R}^{l_1}$, and $h_2 : (\mathbf{R}^n)^2 \times (\mathbf{R}^m)^2 \times [0,T]^2 \to \mathbf{R}^{l_2}$ satisfy

$$\begin{cases} f_1(w,w,x,x,y,y,z_1,t) = \tilde{f}_1(w,x,y,z_1,t), \\ f_2(w,w,x,x,y,y,z_2,t) = \tilde{f}_2(w,x,y,z_2,t), \end{cases} \tag{4}$$

and

$$\begin{cases} h_1(x,x,y,y,s,t) = \tilde{h}_1(x,y,s,t), \\ h_2(x,x,y,y,s,t) = \tilde{h}_2(x,y,s,t), \end{cases} \tag{5}$$

where $w, x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, $z_1 \in \mathbf{R}^{l_1}$, $z_2 \in \mathbf{R}^{l_2}$, and $s, t \in [0,T]$. The above splitting functions are often adopted as Jacobi or Gauss-Seidel.

We also study in this paper the well-known Picard iteration for (1). This kind of iterations is a special form of WR in (3), namely

$$\begin{cases} \dot{x}^{(k+1)}(t) &=& \tilde{f}_1(\dot{x}^{(k)}(t), x^{(k)}(t), y^{(k)}(t), \int_0^t \tilde{h}_1(x^{(k)}(s), y^{(k)}(s), s, t)ds, t), \\ y^{(k+1)}(t) &=& \tilde{f}_2(\dot{x}^{(k)}(t), x^{(k)}(t), y^{(k)}(t), \int_0^t \tilde{h}_2(x^{(k)}(s), y^{(k)}(s), s, t)ds, t), \\ & & x^{(k+1)}(0) = x_0, \quad t \in [0,T], \quad k = 0, 1, \cdots, \end{cases} \tag{6}$$

where $[x^{(0)}(\cdot), y^{(0)}(\cdot)]^t$ is an initial guess as before.

It is known that a circuit system with lumped elements may have the form of (1). For example, if all the elements of a circuit are linear we can then describe the circuit by a system of linear IDAEs [7]. For a high-speed integrated circuit, its equation form may be written as nonlinear differential-algebraic equations (DAEs) with multiple delays if the transmission lines are lossless [8]. As long as the distributed elements (R, L, C, and G) exist in a large circuit we will meet nonlinear IDAEs with multiple delays in the time-domain simulation, see [9]. In other words, some complex differential and integral equations are often arising in the modern circuit simulation field. Here, we will not concretely concern the modelling problems which are really beyond the scope of the paper.

As a simple case, namely without the term $y(\cdot)$ and the second part of (1), a discrete-time WR version is considered in [10]. Moreover, WR solutions of Volterra integral equations are studied in [11]. To take advantage of Lipschitz constants of the nonlinear functions in a system of DAEs, WR is successfully applied to compute their numerical solutions [12, 13]. By a different approach from the known ones we have presented a general convergence condition about continuous-time WR solutions of nonlinear DAEs in [14]. The interesting approach can easily treat complex systems with WR decoupling. It is the first time that the spectral approach is used in WR solutions of nonlinear IDAEs.

In this paper we mainly study the convergence conditions of the continuous-time WR algorithm (3) and the Picard iteration (6) based on operations of linear operators and spectral analysis in function space. We also discuss discrete-time WR solutions by a backward-differentiation formula (BDF). Some typical systems with WR are further included into the paper. Numerical experiments on a test example are provided to illustrate these new convergence conditions.

2. Continuous-time Waveform Relaxation

First we assume that the splitting function pairs (f_1, f_2) and (h_1, h_2) respectively satisfy the following Lipschitz conditions.

Condition (L_f). For four vector norms $\|\cdot\|_n$ in $\mathbf{R}^n$, $\|\cdot\|_m$ in $\mathbf{R}^m$, and $\|\cdot\|_{l_i}$ in $\mathbf{R}^{l_i}$ $(i = 1, 2)$, we assume that there are constants $a_i, b_i (i = 1, 2, \cdots, 6), \alpha$, and β such that

$$\begin{array}{ll} & \|f_1(u_1, u_2, \cdots, u_6, z_1, t) - f_1(v_1, v_2, \cdots, v_6, w_1, t)\|_n \\ \leq & \sum_{i=1}^4 a_i \|u_i - v_i\|_n + \sum_{i=5}^6 a_i \|u_i - v_i\|_m + \alpha \|z_1 - w_1\|_{l_1}, \end{array} \tag{7}$$