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UNIFORM SUPERAPPROXIMATION OF THE DERIVATIVE OF TETRAHEDRAL QUADRATIC FINITE ELEMENT APPROXIMATION *1)

Jing-hong Liu
(*Institute of Applied Mathematics, Hunan University, Changsha 410082, China*)
(*School of Mathematics and Computer Science, Hunan Normal University, Changsha 410081, China*)

Qi-ding Zhu
(*School of Mathematics and Computer Science, Hunan Normal University, Changsha 410081, China*)

Abstract

In this paper,we will prove the derivative of tetrahedral quadratic finite element approximation is superapproximate to the derivative of the quadratic Lagrange interpolant of the exact solution in the L^∞-norm, which can be used to enhance the accuracy of the derivative of tetrahedral quadratic finite element approximation to the derivative of the exact solution.

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1. Introduction

Recently, J.H. Brandts and M. Křížek [1] discussed the superconvergence of tetrahedral quadratic finite elements. Their work focused on the superapproximation of the gradient of the quadratic finite element approximation to the gradient of the quadratic Lagrange interpolant of the exact solution in L^2-norm. For the same model problem, utilizing the theory of the discrete Green's function, this paper studies the superapproximation in L^∞-norm.

2. Preliminaries

Let Ω be a convex bounded polyhedral domain in R^3 with Lipschitz boundary and denote by $W^{k,p}(\Omega)$ the usual Sobolev spaces of functions having generalized partial derivatives up to order k in $L^p(\Omega)$ and their usual norm and seminorm by $\|\cdot\|_{k,p}$ and $|\cdot|_{k,p}$, respectively. In addition, we denote by $W_0^{1,p}(\Omega)$ the subspace of $W^{1,p}(\Omega)$ with $\operatorname{supp} u \subset \Omega$ for each $u \in W_0^{1,p}(\Omega)$. In particular, we set

$$H^k(\Omega) = W^{k,2}(\Omega), \qquad H_0^1(\Omega) = W_0^{1,2}(\Omega)$$
$$\|\cdot\|_k = \|\cdot\|_{k,2}, \qquad |\cdot|_k = |\cdot|_{k,2}.$$

In this paper, let $\mathcal{T}^h$ be the same uniform partition of $\bar{\Omega}$ into tetrahedra as in [1], and h be the largest diameter of all element E from the partition $\mathcal{T}^h$. Relative to the partition $\mathcal{T}^h$, let S_h^k be the k-order finite element subspace of $H^1(\Omega)$, and set $S_{0h}^k = S_h^k \cap H_0^1(\Omega)$. Let $L_h : H^2(\Omega) \to S_h^1$ be the linear Lagrange interpolation operator on the vertices of the tetrahedra, and $Q_h : H^2(\Omega) \to S_h^2$ be the quadratic Lagrange interpolation operator on the vertices and midpoints of edges of the tetrahedra.

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Now we introduce the subspace $B_{0h}^2 \subset S_{0h}^2$ of so-called quadratic bubble functions, defined by

$$B_{0h}^2 = \left\{ (I - L_h)v \mid v \in S_{0h}^2 \right\}.$$

This definition induces the following space-decomposition

$$S_{0h}^2 = S_{0h}^1 \oplus B_{0h}^2,$$

which expresses that each $v \in S_{0h}^2$ can be uniquely written as $l + b$ with $l \in S_{0h}^1$ and $b \in B_{0h}^2$ (cf. [1]). This decomposition will be used in our main results. Obviously, B_{0h}^2 is spanned by the basis ψ_i, $(i = 1, \cdots, M)$, where each $\psi_i \in S_{0h}^2$ has a positive value at the midpoint of the internal edge e_i, has norm $|\psi_i|_1 = 1$, and vanishes at all other edges.

Next, we define *discrete* δ function $\delta_z^h \in S_{0h}^2(\Omega)$, *discrete derivative* δ function $\partial_z \delta_z^h \in S_{0h}^2(\Omega)$, L^2 *projection* $Pu \in S_{0h}^2(\Omega)$ of $u \in L^2(\Omega)$, *discrete derivative* Green's function $\partial_z G_z^h \in S_{0h}^2(\Omega)$, and *derivative zhun* Green's function $\partial_z G_z^* \in H_0^1(\Omega)$ as follows [2]:

$$\begin{aligned}
(v,\, \delta_z^h) &= v(z), && \forall v \in S_{0h}^2(\Omega)\\
(u - Pu,\, v) &= 0, && \forall v \in S_{0h}^2(\Omega)\\
(v,\, \partial_z \delta_z^h) &= \partial v(z), && \forall v \in S_{0h}^2(\Omega)\\
(\nabla \partial_z G_z^h,\, \nabla v) &= \partial v(z), && \forall v \in S_{0h}^2(\Omega)\\
(\nabla \partial_z G_z^*,\, \nabla v) &= (\partial_z \delta_z^h,\, v), && \forall v \in H_0^1(\Omega)
\end{aligned}$$

where $S_{0h}^2(\Omega) \subset H_0^1(\Omega)$ is the quadratic tetrahedral finite element space. Obviously, $\partial_z G_z^h$ is the finite element approximation to $\partial_z G_z^*$.

In addition, for $u \in H_0^1(\Omega)$, we can easily obtain

$$(\nabla \partial_z G_z^*,\, \nabla u) = (\partial_z \delta_z^h,\, u) = (\partial_z \delta_z^h,\, Pu) = \partial_z Pu(z).$$

Further, the following stability estimate holds

$$\|Pu\|_{1,\,q} \le C \|u\|_{1,\,q} \quad for \;\; 3 < q \le \infty,$$

which can be similarly proved as Corollary 2 in Zhu,Lin[2, pp104].

Finally, we will give the following two fundamental assumptions which are needed in next sections (cf. [2, 3]):

(A1). For the model problem (1) considered in Section 3, there exist $1 < q_0 \le \infty$ and a constant $C(p)$ such that the following a priori estimate holds

$$\|u\|_{2,\,p,\,\Omega} \le C(p) \|f\|_{0,\,p,\,\Omega},\ \forall\, 1 < p < q_0,\ u \in W^{2,\,p}(\Omega) \cap W_0^{1,\,p}(\Omega).$$

(A2). For each $v \in W^{2,\,q}(\Omega) \cap W_0^{1,\,q}(\Omega)$ there exists a $\chi \in S_{0h}^2$ such that

$$\|v - \chi\|_{1,\,q} \le Ch \|v\|_{2,\,q} \quad for \; 1 \le q \le \infty.$$

In this paper we shall use letter C to denote a generic constant which may not be the same in each occurrence.

3. The Tetrahedral Quadratic Finite Element Method

Let us consider the following boundary value problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$