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EXPECTED NUMBER OF ITERATIONS OF INTERIOR-POINT ALGORITHMS FOR LINEAR PROGRAMMING *1)

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Abstract

We study the behavior of some polynomial interior-point algorithms for solving random linear programming (LP) problems. We show that the expected and anticipated number of iterations of these algorithms is bounded above by $O(n^{1.5})$. The random LP problem is Todd's probabilistic model with the Cauchy distribution.

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1. Introduction

Since Karmarkar [4] introduced his $O(nL)$-iteration projective algorithm, the area of interior point algorithms for linear programming have developed rapidly. Many other algorithms have been introduced to the growing literature on interior point algorithms, for examples, path-following algorithms; potential reduction algorithms; and predictor-corrector algorithms, etc. The best known worst-case iteration complexity for interior point algorithms is $O(\sqrt{n}L)$, where n is the number of variables and L is the input data length of the LP problems.

In practice the interior point algorithms also performed compatetive with simplex algorithm. People (e.g.,Lustig et la. [6], Yang and Huang [10]) have observed that the number of iterations needed to solve the LP problems is $O(\ln n)$ using regression. Therefore there is a gap between the theoretical worst case complexity and practical performance of the interior point algorithms. Ye [11] showed that the anticipated number of iterations of interior point algorithms is bounded above by $O(\sqrt{n}\ln n)$. Recently, Anstreicher et al. [1] have obtained expected number of iterations bound of $O(n \ln n)$ for a variant of degenerate random LP model (Model II of Todd [9]) using the infeasible primal-dual algorithms of Potra [8]. Huang [3] has shown that the expected number of iterations of some feasible interior point algorithms (e.g., Kojima et la. [5]) is bounded above by $O(n^{1.5})$ for a nondegenerate random LP model (Model I of Todd [9]).

In this paper, we will show that the expected and anticipated number of iterations of some interior point algorithms is bounded above by $O(n^{1.5})$ for solving a random LP model which is an extension of Todd's model I in [9].

The paper is organized as follows. In section 2, we introduce the random LP model and some useful results. We review the stopping criterion for polynomial interior-point algorithms in section 3. Section 4 derives a bound for the expected number of iterations for solving the random LP model using certain interior point algorithms. We show the anticipated result in section 5. Finally we will give some concluding remarks in section 6.

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2. The Probabilistic Model

We consider the following probabilistic model which is an extension of Todd's model I in [9].

(LP) minimize $e^T x$
subject to $Ax = b,\ x \geq 0,$

where $b = Ae$ and $e \in R^n$ is a vector of all ones, $A = (a_{ij}) \in R^{m\times n}$ is a random matrix whose entries are independently and identically distributed as Cauchy distribution with characteristic function $e^{-c|t|}$ $(c > 0)$. It's dual form can be stated as:

(LD) minimize $e^T s$
subject to $s = e - A^T y, s \geq 0,$

since $b^T y = (Ae)^T y = e^T A^T y = e^T(e - s) = n - e^T s$. Therefore max $b^T y$ is equivalent to min $e^T s$. The following lemma will help us to derive the distribution of the optimal solution of above random LP model.

Lemma 2.1. *Consider the system $Bx = d$, where $B = (a_{ij}) \in R^{m\times m}$ is a random matrix such that a_{ij} $(i, j = 1, \cdots, m)$ are independent and identical(iid) Cauchy random variables, and $d \in R^m$ is a random vector such that $d_i (i = 1, \cdots, m)$ are iid Cauchy random variables. Assume the columns of B and d are independent. Then the random variables x_k $(k = 1, \cdots, m)$ are distributed as $\frac{\lambda_k}{\lambda_0}$ $(k = 1, \cdots, m)$ where λ_k $(k = 0, 1, \cdots, m)$ are independent and identical random variables with Cauchy distribution.*

Proof. The proof is similar to the proof in Girko [2] where the matrix B is not a square matrix. For completion we include it here. It is easy to see that $detB \neq 0$ with probability one. By Cramer's rule we have

$$x_k = \frac{\sum_{i=1}^m d_i B_{ik}}{\sum_{i=1}^m a_{ik} B_{ik}} = \frac{(\sum_{i=1}^m d_i B_{ik})R^{-1}}{(\sum_{i=1}^m a_{ik} B_{ik})R^{-1}}, \qquad (*)$$

where B_{ik} $(i = 1, \cdots, m)$ is the cofactor of the a_{ik} in the matrix B, $R = (\sum_{i=1}^m |B_{ik}|)$ and $|B_{ik}|$ is the determinant of B_{ik}.

Next we calculate the joint characteristic function of the numerator and denominator of $(*)$ using conditional expectation:

$$\begin{aligned}
&Eexp\{it\sum_{i=1}^m d_i B_{ik} R^{-1} + i\tau \sum_{i=1}^m a_{ik} B_{ik} R^{-1}\} \\
&= EE[exp\{it\sum_{i=1}^m d_i B_{ik} R^{-1} + i\tau \sum_{i=1}^m a_{ik} B_{ik} R^{-1}\}|a_{\nu\mu}, \nu = 1, \cdots, m, \mu = \{1, \cdots, m\}/k] \\
&= E[\prod_{i=1}^m exp\{-c|tB_{ik}|R^{-1}\} \prod_{i=1}^m exp\{-c|\tau B_{ik}|R^{-1}\}] \\
&= exp\{-c|t|\}exp\{-c|\tau|\}.
\end{aligned}$$

Therefore the numerator and denominator are independent and identically distributed as Cauchy distribution.

Using lemma 2.1 we can obtain following two lemmas which discuss the distribution of the vertices of (LP) and (LD).

Lemma 2.2. *Let λ_0, λ_1, $\cdots$, λ_m be independent and identical Cauchy random variables. Then a vertex (may not be feasible) of (LP) has its basic variables distributed like $1 + d\frac{\lambda_i}{\lambda_0}$ for $i = 1, \cdots, m$ $(d = n - m)$ and its nonbasic variables equal 0. Furthermore, every vertex of (LP) is nondegenerate with probability one.*

Proof. Assume that first m columns of A are basic columns and is denoted by B, and