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GENERALIZED LAGUERRE APPROXIMATION AND ITS APPLICATIONS TO EXTERIOR PROBLEMS *1)

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Abstract

Approximations using the generalized Laguerre polynomials are investigated in this paper. Error estimates for various orthogonal projections are established. These estimates generalize and improve previously published results on the Laguerre approximations. As an example of applications, a mixed Laguerre-Fourier spectral method for the Helmholtz equation in an exterior domain is analyzed and implemented. The proposed method enjoys optimal error estimates, and with suitable basis functions, leads to a sparse and symmetric linear system.

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1. Introduction

Many practical problems in science and engineering require solving partial differential equations in exterior domains. Considerable progress has been made recently in using spectral methods for solving partial differential equations in unbounded domains. The first approach is based on the classical orthogonal systems in the unbounded domains, namely, the Hermite (cf. [7, 12, 10]) and Laguerre (cf. [16, 6, 17, 14, 18, 19, 20]) polynomials/functions. The second approach is to map the original problem in a unbounded domain to a singular problem in a bounded domain (cf. [8, 11, 13]). The third approach is based on rational approximations (cf. [3, 2, 5, 15, 9]). However, none of the methods mentioned above has yet been analyzed for multidimensional exterior problems.

In this paper, we investigate the spectral approximation using generalized Laguerre polynomials which form a mutually orthogonal system in the weighted Sobolev space $L^2_{\omega_\alpha}(0,\infty)$ with $\omega_\alpha(\rho) = \rho^\alpha \exp(-\rho)$. The orthogonal projection in $L^2_{\omega_\alpha}(0,\infty)$ has been analyzed in [6]. Other projection and interpolation operators for the special case $\alpha = 0$ have been studied in [16, 17, 14, 20]. However, the usual weighted Sobolev spaces used in these papers are not the most appropriate. Here, we study the generalized Laguerre approximations in non-uniformly weighted spaces, i.e., with different weights for derivatives of different orders, and we obtain

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optimal results for several projection operators for all $\alpha > -1$. These new results enable us to study numerical approximations of a large class of problems in unbounded domains.

As an example of applications, we consider the Helmholtz equation in the two dimensional exterior domain $\Omega = \{(\rho,\theta) : \rho > 1,\ \theta \in [0,2\pi)\}$. We propose a mixed Laguerre-Fourier spectral method using Laguerre polynomials for the radial direction and Fourier series for the azimuthal direction. Thanks to the new results on generalized Laguerre approximations, we are able to prove optimal error estimates for the mixed Laguerre-Fourier method applied to the transformed equation. Furthermore, by choosing a set of suitable basis functions, we are also able to construct an efficient numerical algorithm in which the linear system is symmetric and sparse, and hence can be efficiently solved.

The paper is organized as follows. In the next section, we present several basic approximation results using generalized Laguerre polynomials. Then, we study the mixed Laguerre-Fourier approximation outside a disk in Section 3. We construct the mixed Laguerre-Fourier spectral scheme for a model problem, and prove its convergence in Section 4. In Section 5, we present implementation details and an illustrative numerical result. Some concluding remarks are presented in the final section.

2. Generalized Laguerre Approximation

2.1 Notations and preliminaries

Let us first introduce some notations. Let $\Lambda = \{\rho \mid 0 < \rho < \infty\}$ and $\chi(\rho)$ be a certain weight function in the usual sense. We define

$$L^2_\chi(\Lambda) = \{\ v \mid v \ \text{ is measurable on } \Lambda \text{ and } \|v\|_{L^2_{\chi,\Lambda}} < \infty\ \}$$

with the following inner product and norm,

$$(u,v)_{\chi,\Lambda} = \int_\Lambda u(\rho)v(\rho)\chi(\rho)d\rho, \qquad \|v\|_{\chi,\Lambda} = (v,v)^{\frac{1}{2}}_{\chi,\Lambda}.$$

For simplicity, we denote by $\partial^k_\rho v$ the k-th derivative of $v(\rho)$ with respect to ρ. For any non-negative integer m, we define the weighted Sobolev space

$$H^m_\chi(\Lambda) = \left\{v \mid \partial^k_\rho v \in L^2_\chi(\Lambda),\ 0 \le k \le m\right\}$$

equipped with the following inner product, semi-norm and norm

$$(u,v)_{m,\chi,\Lambda} = \sum_{0\le k\le m} (\partial^k_\rho u, \partial^k_\rho v)_{\chi,\Lambda}, \quad |v|_{m,\chi,\Lambda} = \|\partial^m_\rho v\|_{\chi,\Lambda}, \quad \|v\|_{m,\chi,\Lambda} = (v,v)^{\frac{1}{2}}_{m,\chi,\Lambda}.$$

For any real $r > 0$, the space $H^r_\chi(\Lambda)$ and its norm $\|v\|_{r,\chi,\Lambda}$ are defined by space interpolation as in Adams [1]. In particular, we denote

$${}_0H^1_\chi(\Lambda) = \{v \mid v \in H^1_\chi(\Lambda) \text{ and } v(0) = 0\}.$$

Let $\omega_\alpha(\rho) = \rho^\alpha e^{-\rho}$. We denote in particular $\omega(\rho) = \omega_0(\rho) = e^{-\rho}$. The generalized Laguerre polynomials of degree l are defined by

$$\mathcal{L}^{(\alpha)}_l(\rho) = \frac{1}{l!}\rho^{-\alpha}e^\rho \partial^l_\rho(\rho^{l+\alpha}e^{-\rho}), \qquad l = 0,1,2,\cdots,\ \alpha > -1.$$

They are eigenfunctions of the Sturm-Liouville problem

$$\partial_\rho(\omega_{\alpha+1}(\rho)\partial_\rho v(\rho)) + \lambda\omega_\alpha(\rho)v(\rho) = 0, \qquad 0 < \rho < \infty, \tag{2.1}$$