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EXPONENTIAL MESH APPROXIMATIONS FOR A 3D EXTERIOR PROBLEM IN MAGNETIC INDUCTION *

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Dedicated to the Memory of Doctor E. Nabana

Abstract

A numerical method combining the approaches of C.I. Goldstein and L.-A. Ying is used for the simulation in three-dimensional magnetostatics related to an exterior problem in magnetic induction. Recently introduced, this method is based on the use of a graded mesh obtained by gluing homothetic layers in the exterior domain and has been performed in the case of edge element discretizations. In this work, the theoretical and practical aspects of the method are inspected in the case of face element and volume element discretizations, for computing a magnetic induction. Error estimates, implementations, and numerical results are provided.

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Key words: Exterior problems, Magnetostatics, Mixed formulations, Graded meshes, Face elements, Volume elements, Boundary elements, Truncations, Error estimates.

1. Introduction

When we are concerned with the numerical simulation associated with a linear exterior problem, we use in major cases the boundary integral method for discretizing the problem. This approach requires the discretization of boundary integral operators and following the usual processes, we are led to consider dense matrices in computations. Typically, when we compute the magnetic induction (see e.g. [15]) in three-dimensional magnetostatics, we use a vector-valued boundary integral operator with which a dense matrix is associated by finite element techniques. The assembling of such a matrix is not easy, and moreover its size, proportional to the square of the number of boundary edges, can forbid fine meshes for storage requirements.

Goldstein's approach (see [14]) is an alternative to the boundary integral method. This approach is based on the coupling of the finite element method with an appropriately graded mesh near infinity. The original exterior problem is rewritten in a bounded truncated domain for which the boundary is near infinity. The definition of this truncated domain and the use of a graded mesh are crucial in such a way that optimal error estimates hold between the original continuous solution and the discrete solution—resulting from the truncated domain.

In another approach, Ying has introduced in [20] an infinite mesh method for exterior domains. The method is based on a superposition of homothetic layers and therefore provides a kind of graded mesh. The main difference between the boundary integral method and Ying's approach concerns the discretization of boundary integral operators. Namely in the case of the Poincaré-Steklov operator, he builds recursively a sequence of stiffness matrices that converges to the stiffness matrix of the infinite mesh corresponding thus to the discretization of the Poincaré-Steklov operator.

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Other alternatives to the boundary integral method can be found in [4], [13], [16] as well as in additional references cited therein.

We are concerned here with an approach recently introduced in [1], [2], called the *exponential mesh approximation*, mixing the methods of Goldstein [14] and Ying [20], and which consists of building an infinite mesh as in [20] and of doing truncations of this mesh as in [14]. This method has been applied to the computation of the demagnetizing potential in micromagnetics [1], and to the computation of a magnetic reaction field [2]. The exponential mesh consists of an assembling of homothetic layers in the exterior domain and gives a natural way to get a graded mesh at infinity. Locally in each homothetic layer, finite element approximations are considered. For example, Lagrange's elements are used in [1] and edge elements are used in [2]. Here, we will consider face element and volume element approximations.

The exterior problem considered hereafter comes from magnetostatics and consists of finding a vector field B such that:

$$\operatorname{div} B \,=\, 0 \;\text{ in }\; \mathbb{R}^3 \;\text{ and }\; \operatorname{curl}(\nu\, B) \,=\, J \;\text{ in }\; \mathbb{R}^3\,.$$

The datum J is a current density, B is the magnetic induction, and ν is a physical parameter used to describe the magnetic reluctance of the considered material. In what follows, a magnetic material will typically be represented by a bounded domain Ω with boundary Γ. Also, ν will take in $\Omega' = \mathbb{R}^3 \setminus \overline{\Omega}$ the value ν_0, the reluctance of the vacuum. Time-independent, the current density is considered here as: $J = \widetilde{j}$, the extension, outside Ω by zero, of a vector field j confined to Ω, divergence-free, square integrable with normal trace on Γ equal to zero. In the space, this current density creates a source field h^s: $\operatorname{curl} h^s = J$, $\operatorname{div} h^s = 0$ in $\mathbb{R}^3$, which can be explicitly determined with the help of the Biot-Savart formula [5]. The original system is thus reformulated as a new problem where h^s appears as a datum: find a vector field B such that

$$\operatorname{div} B \;=\; 0 \;\text{ in }\; \mathbb{R}^3\,, \tag{1}$$

$$\operatorname{curl}(\nu\, B) \;=\; \operatorname{curl} h^s \;\text{ in }\; \mathbb{R}^3\,, \tag{2}$$

$$\lim_{|x|\to\infty} |B(x)| \;=\; 0\,.$$

A recent approach (see e.g. [15]) proposed for solving $(1)-(2)$ consists of considering a mixed formulation in which the restriction of B to Ω, used as unknown in Ω, is coupled on Γ with a vector-valued boundary unknown allowing to represent B in Ω' with the help of a vector potential. With such a formulation, which uses of course a vector-valued boundary integral operator, $(1)-(2)$ is treated by a mixed finite element method.

A variational mixed method is also used in our approach in order to solve $(1)-(2)$. Namely, besides B which appears as an unknown in the considered mixed formulation, a scalar field also defined in $\mathbb{R}^3$ is used as an auxiliary unknown. A first difference with the mixed formulation using a boundary integral operator is that B is no longer represented in Ω' with the help of a vector potential. More precisely, our formulation does not use any boundary unknown and we suggest to discretize this formulation with the help of exponential mesh approximations.

This work contains five sections. In section 2 we consider some notations and introduce from $(1)-(2)$ a mixed formulation in magnetic induction.

Exponential mesh approximations are reported in section 3. In this part, we start by describing the discretization of the whole space $\mathbb{R}^3$ with the help of an exponential mesh. Then, we introduce the discrete spaces of infinite dimension in which the unknowns are determined, and consider a discrete formulation on the exponential mesh for which we derive an error estimate. This formulation yields a discrete system of infinite dimension and we truncate the exponential mesh in order to reduce the size of the system to a finite dimension. A second error estimate for the truncated system is established with an asymptotic formula between the interior mesh size, the homothetic coefficient ξ and the number of homothetic layers N considered for the exponential mesh. We consider two kinds of boundary condition on the magnetic induction