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ON CONVERGENCE OF MULTIGRID METHOD FOR NONNEGATIVE DEFINITE SYSTEMS *1)

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Abstract

In this paper, we consider multigrid methods for solving symmetric nonnegative definite matrix equations. We present some interesting features of the multigrid method and prove that the method is convergent in L_2 space and the convergent solution is unique for such nonnegative definite system and given initial guess.

Mathematics subject classification: 65N55, 65F10, 65N30.
Key words: Multigrid, Singular Problem, Convergence.

1. Introduction

Multigrid (MG) methods have been successfully applied to many scientific computing problems. The main advantage of this method is its asymptotically optimal convergence, *i.e.*, the computational work required to achieve a fixed accuracy is proportional to the number of discrete unknowns. The convergence analysis of multigrid methods has been studied extensively by many papers (see [3, 6, 9, 11, 12, 16, 17, 18]). Recent effort for indefinite systems has been made in [5, 8, 21].

In this paper, we consider convergence of the multigrid method for linear systems with symmetric nonnegative definite matrices. Classical iterative algorithms, such as Jacobi iteration and Gauss-Seidel iteration, for solving such nonnegative definite systems have been well studied in many literatures (*e.g.*, see [1]). Some semiconvergent iterative methods were discussed in [7,13]. An incomplete factorization and an extrapolation technique were presented in [15] and [19], respectively. The convergence analysis of these classical iterations for the semidefinite problems can be obtained due to simple structures of algorithms. It has been proved theoretically and numerically that multigrid methods are usually more efficient than those classical iterations. Some numerical investigation of multigrid methods has been presented for solving certain singular systems arising from eigenproblems, second-order elliptic PDEs with Neumann boundary conditions, queuing networks, and image reconstruction ([3,5,10]). Theoretical analysis for the indefinite systems is less explored. The major difficulties lie on the fact that there exist infinite many solutions for a consistent singular system and the structure of multigrid methods is more complicated than those of classical iterations. The concept of classical convergence should be modified. In fact, for a singular system, one only expects to find an approximation to one of solutions. In this case, the main point for an iterative algorithm is as follows: when the iteration stops, the difference between the iterative solution and some exact solution is less than a given tolerance. In this paper, we shall prove that multigrid methods for symmetric nonnegative

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definite systems are convergent in a classical sense (L_2-norm). Some important features will be discussed. We present a convergence rate in a quotient space (in an energy norm) and an asymptotic convergence rate in the classical sense (L_2-norm). Thus, the multigrid method, similar to some classical iterations, is a semiconvergent method and can be applied directly to symmetric nonnegative systems of equations.

The paper is organized as follows. We present a multigrid algorithm and some new features for the singular case in section 2. The general convergence theory of multigrid methods for semidefinite problems are discussed in section 3.

2. Multigrid Algorithm

Let V^m, $m = 1, 2, ..., M$, be nested n_m dimensional Hilbert spaces with inner product $(\cdot, \cdot)$ and $\|v^m\|_{L_2} = (v^m, \quad v^m)^{1/2}$. Let $A^m \geq 0$ be an $n_m \times n_m$ symmetric nonnegative definite matrix on V^m with null space $N(A^m)$. Denote the quotient space $V^m/N(A^m)$ by H^m. Then $A^m > 0$ on H^m and $H^m = span\{v_1^m, v_2^m, ..., v_{l_m}^m\}$, where v_l^m, $1 \leq l \leq l_m$, are the eigenvectors corresponding to nonzero eigenvalues and l_m is the rank of the matrix A^m.Let $P_H^m : V^m \to H^m$ and $P_0^m : V^m \to N(A^m)$ be the orthogonal projection operators. For any $v \in V^m$, we have $v = v_H + v_0$, where $v_H = P_H^m v$ and $v_0 = P_0^m v$.

Consider the matrix problem

$$A^1 U^1 = R^1, \tag{2.1}$$

and assume that A^1 is irreducible and symmetric nonnegative definite and the right-hand side R^1 is given properly such that there exists at least a solution for the above problem (2.1), *i.e.*, R^1 is in the quotient space H^1.

Let $I_m^{m+1} : V^m \to V^{m+1}$ define a restriction and $I_{m+1}^m : V^{m+1} \to V^m$ an interpolation, $1 \leq m \leq M-1$. Let G^m: $V^m \times V^m \to V^m$ be smoothing operators and F^M represents an exact solver, in which case $F^M(U^M, R^M) = U^{M,*}$, where $A^M U^{M,*} = R^M$. The following defines a standard μ-cycle multigrid algorithm (called a V-cycle if $\mu = 1$ and a W-cycle if $\mu = 2$) for solving

$$A^m U^m = R^m, \quad 1 \leq m \leq M. \tag{2.2}$$

Standard MG Algorithm

(i) If $m = M$, then $U^M \leftarrow F^M(U^M, R^M)$.

(ii) If $m < M$, then

(1) $U^m \leftarrow G^m(U^m, R^m)$ (pre-smoothing step);

(2) perform μ iterations of Standard MG Algorithm on level $m+1$ (with fixed value of U^m) for the following correction problem, starting from zero initial value :

$$A^{m+1} U^{m+1} = I_m^{m+1}(R^m - A^m U^m), \quad U^{m+1} \in V^{m+1}\,; \tag{2.3}$$

(3) $U^m \leftarrow U^m + I_{m+1}^m U^{m+1}$ (correction);

(4) $U^m \leftarrow G^m(U^m, R^m)$ (post-smoothing step).

Here we assume that the interpolation I_{m+1}^m is full rank and

$$I_m^{m+1} = (I_{m+1}^m)^T \quad \text{and} \quad A^{m+1} = I_m^{m+1} A^m I_{m+1}^m. \tag{2.4}$$

It has been noted that the standard MG algorithm is given in a recurrence form. The matrix A^{m+1} is irreducible and symmetric nonnegative definite if A^m possesses these properties. More important features are given in the following lemma.

Lemma 2.1. *Assume that A^m is irreducible and symmetric nonnegative definite, and the interpolation operator I_{m+1}^m is full rank. Then, we have*

$$I_{m+1}^m : N(A^{m+1}) \to N(A^m),$$

and

$$I_m^{m+1} : H^m \to H^{m+1}.$$