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NON C^0 NONCONFORMING ELEMENTS FOR ELLIPTIC FOURTH ORDER SINGULAR PERTURBATION PROBLEM *1)

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Abstract

In this paper we give a convergence theorem for non C^0 nonconforming finite element to solve the elliptic fourth order singular perturbation problem. Two such kind of elements, a nine parameter triangular element and a twelve parameter rectangular element both with double set parameters, are presented. The convergence and numerical results of the two elements are given.

Mathematics subject classification: 65N12, 65N30.
Key words: Singular perturbation problem, Nonconforming element, Double set parameter method.

1. Introduction

We consider the following elliptic singular perturbation problem [1]:

$$\begin{cases} \varepsilon^2\Delta^2 u - \Delta u = f & \text{in } \Omega \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $f \in L^2(\Omega), \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator, $\Delta^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})^2$, $\Omega \subset R^2$ is a bounded polygonal domain, $\partial\Omega$ is the boundary of Ω, $\frac{\partial}{\partial n}$ denotes the outer normal derivative on $\partial\Omega$, and ε is a real parameter such that $0 < \varepsilon \le 1$. When ε tends to zero, (1) formally degenerates to Poisson's equation. Hence, (1) is a plate model which may degenerate toward an elastic membrane problem.

A conforming plate element should have C^1 continuity which makes the element complicated, so nonconforming plate elements are widely used. For convergence criterion there are Patch-Test[10] which is convenient to use for engineers, and Generalized Patch-Test[9] which is a sufficient and necessary condition. According to Generalized Patch-Test, Professor Shi presented F-E-M-Test[11] which is easier to use. Many successful nonconforming plate elements [5,7,3,12,13,14] have been presented, but not all of them are convergent for (1) uniformly respect to ε.

It is proved[1] that the non-C^0 nonconforming plate element— Morley's element [2],—is not convergent for (1) when $\varepsilon \to 0$. In [1] a C^0 nonconforming plate element is presented, which is convergent for (1) uniformly in ε. In this paper we study the convergence of non-C^0 nonconforming plate elements for (1). In section 2 we give a general convergence theorem for non-C^0 nonconforming plate elements solving (1). In section 3 the double set parameter method to construct nonconforming finite element is presented. In section 4 a triangular and a rectangular non-C^0 nonconforming plate elements [3][4] are presented and their convergence for (1) uniformly in ε is proved. In section 5 some numerical results are given.

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2. A Convergence Theorem

The inner product on $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$, $H^m(\Omega)$ is the usual Sobolev space of functions with partial derivatives of order less than or equal to m in $L^2(\Omega)$, and the corresponding norm by $\|\cdot\|_{m,\Omega}$. The seminorm derived from the partial derivatives of order equal to m is denoted by $|\cdot|_{m,\Omega}$. The space $H_0^m(\Omega)$ is the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. Alternatively, we have

$$H_0^1(\Omega)=\{v\in H^1(\Omega); v|_{\partial\Omega}=0\}, H_0^2(\Omega)=\{v\in H^2(\Omega); v=\frac{\partial v}{\partial n}=0, on \partial\Omega\}$$

Let Du be the gradient of u and $D^2u=(\frac{\partial^2 u}{\partial x_i\partial x_j})_{2\times 2}$ be the 2×2 tensor of the second order partial derivatives.

The weak form of (1) is : find $u\in H_0^2(\Omega)$ such that

$$\varepsilon^2 a(u,v)+b(u,v)=(f,v) \qquad \forall v\in H_0^2(\Omega) \tag{2}$$

where

$$a(u,v)=\int_\Omega D^2u:D^2v dx, \qquad b(u,v)=\int_\Omega Du\cdot Dv dx. \tag{3}$$

From Green's formula[5], it is easy to see that

$$\int_\Omega D^2u:D^2v dx=\int_\Omega \triangle u\triangle v dx \qquad \forall u,v\in H_0^2(\Omega) \tag{4}$$

However this identity does not hold on the noncomforming finite element spaces. We use the form (3) like in [1].

Assume that $\{T_h\}$ is a quasi-uniform [5] and shape-regular[5] family of triangulations of Ω, here the discretization parameter h is a characteristic diameter of the elements in T_h. We use V_h to denote the finite element space which is piecewise polynomial space and satisfies the boundary conditions of (1) in some way. Then the finite element approximation of (2) is: find $u_h\in V_h$ such that

$$\varepsilon^2 a_h(u_h,v_h)+b_h(u_h,v_h)=(f,v_h) \qquad \forall v_h\in V_h \tag{5}$$

where

$$a_h(u,v)=\sum_{K\in T_h}\int_K D^2u:D^2v dx, \qquad b_h(u,v)=\sum_{K\in T_h}\int_K Du\cdot Dv dx.$$

We define a seminorm $|||\cdot|||_{\varepsilon,h}$ by [1]

$$|||w|||_{\varepsilon,h}^2=\varepsilon^2 a_h(w,w)+b_h(w,w)=\varepsilon^2|w|_{2,h}^2+|w|_{1,h}^2 \tag{6}$$

where $|\cdot|_{i,h}^2=\sum_K|\cdot|_{i,K}^2, i=1,2.$

The interpolation operator derived by V_h is denoted by Π_h. Let $\Pi_K=\Pi_h|_K$ for $K\in\mathcal{T}_h$. $P_m(K)$ is the polynomial space of degree less than or equal to m on K. Let F denote any edge of an element.

Theorem 1. *Let u and u_h be solutions of (2)and (5) respectively. If V_h satisfies the following conditions:*
(c1) $|||\cdot|||_{\varepsilon,h}$ is a norm on V_h.
(c2) $\forall K\in T_h, \forall v\in P_2(K), \Pi_K v=v$.
(c3) $\forall v_h\in V_h, v_h$ is continuous at the vertics of elements and is zero at the vertics on $\partial\Omega$.
(c4) $\forall v_h\in V_h, \int_F v_h ds$ is continuous across the element edge F and is zero on $F\subset\partial\Omega$.
(c5) $\forall v_h\in V_h, \int_F \frac{\partial v_h}{\partial n}ds$ is continuous across the element edge F and is zero on $F\subset\partial\Omega$.
Then

$$|||u-u_h|||_{\varepsilon,h}\le ch(\varepsilon|u|_{3,\Omega}+|u|_{2,\Omega}+\|f\|_{0,\Omega}) \tag{7}$$