Journal of Computational Mathematics, Vol.23, No.2, 2005, 199–204.

NEW ESTIMATES FOR SINGULAR VALUES OF A MATRIX *

Ming-xian Pang
(*Normal Faculty of Science, Jilin Beihua University, Jilin 132013, China*)

Abstract

New estimates are provided for singular values of a matrix in this paper. These results generalize and improve corresponding estimates for singular values in [4]-[6].

Mathematics subject classification: 15A18, 65F20.
Key words: Singular value, Directed graph, Circuit.

1. Introduction and Denotations

In terms of matrix entries, Gerschgorin's theorem, Brauer's theorem and Brualdi's theorem provide useful estimates for eigenvalues of a matrix ([9],[10]). Using these theorems, some researchers made many corresponding estimates for singular values of a matrix (see [1]-[8]). In this paper, several new estimates for singular values of a matrix are presented. These results generalize and improve corresponding estimates in [4]-[6].

The set of all $n\times n$ complex matrices is denoted by $C^{n\times n}$. Let $A=(a_{ij})\in C^{n\times n}$, $\sigma(A)$ be the set of all singular values of A, and

$$r_i(A)=\sum_{j\neq i}|a_{ij}|,\qquad c_i(A)=\sum_{j\neq i}|a_{ji}|,\qquad a_i=|a_{ii}|,\qquad i\in\langle n\rangle=\{1,2,...,n\}.$$

Suppose the partition $N_j\subseteq\langle n\rangle,\ j\in\langle m\rangle$. Then it satisfies that $\bigcup\limits_{j\in\langle m\rangle}N_j=\langle n\rangle$, and for $\forall\ i\neq j$, $N_i\cap N_j=\phi$. For all $i\in\langle n\rangle$,denote $i\in N_{\sigma_i},\ \sigma_i\in\langle m\rangle$, and let $(\sigma_1,...,\sigma_m)$ be a permutation of $(1,...,m)$. For the sake of convenient, we also use the following denotations:

$$r^{(i)}_{N_{\sigma_i}}(A)=\sum_{j\in N_{\sigma_i}\backslash\{i\}}|a_{ij}|,\qquad c^{(i)}_{N_{\sigma_i}}(A)=\sum_{j\in N_{\sigma_i}\backslash\{i\}}|a_{ji}|;$$
$$\bar r^{(i)}_{N_{\sigma_i(A)}}=r_i(A)-r^{(i)}_{N_{\sigma_i}}(A),\qquad \bar c^{(i)}_{N_{\sigma_i}}(A)=c_i(A)-c^{(i)}_{N_{\sigma_i}}(A);$$
$$S^{(i)}_{N_{\sigma_i}}(A)=\max\{r^{(i)}_{N_{\sigma_i}}(A),\ c^{(i)}_{N_{\sigma_i}}(A)\},\qquad \bar S^{(i)}_{N_{\sigma_i}}(A)=\max\{\bar r^{(i)}_{N_{\sigma_i}}(A),\ \bar c^{(i)}_{N_{\sigma_i}}(A)\}.$$

Let $\Gamma(A)$ be the directed graph of A with vertex set $V=\langle n\rangle$ and $E=\{(i,j):\ a_{ij}\neq 0\}$. The sets of out-neighbors and in-neighbors of i in $\Gamma(A)$ are denoted by $\Gamma^+_i(A)$ and $\Gamma^-_i(A)$, respectively, namely,

$$\Gamma^+_i(A)=\{j\in V\backslash\{i\}:\ (i,j)\in E\},\quad \Gamma^-_i(A)=\{j\in V\backslash\{i\}:(j,i)\in E\}.$$

For a given $A=(a_{ij})\in C^{n\times n}$, we define the undirected graph $G(A)=(\tilde V,\tilde E)$ with vertex set $\tilde V=\langle n\rangle$ and edge set $\tilde E=\{\{i,j\}:\ a_{ij}\neq 0\ or\ a_{ji}\neq 0,\ 1\leq i\neq j\leq n\}$, and denote $G_i(A)=\Gamma^+_i(A)\cup\Gamma^-_i(A),\ E_\sigma=\{\{i,j\}\in\tilde E:\ i\in N_{\sigma_i},\ j\in N_{\sigma_j},\ \sigma_i\neq\sigma_j\}$.

2. Main Results

In this section we give an improved Brauer-type estimate for singular values .

* Received August 25, 2003; final revised November 6, 2003.

Lemma 2.1. *Let* $A=(a_{ij})\in C^{n\times n}$ *and give a partition* $\langle n\rangle=\bigcup\limits_{j\in\langle m\rangle}N_j,\ N_i\cap N_j=\phi,\ i,j\in\langle m\rangle,\ i\neq j$. *If* $G_i(A)\neq\phi,\ \forall i\in\langle n\rangle$ *and* $G_i(A)\cap N_{\sigma_j}\supseteq G_j(A)\cap N_{\sigma_j},\ \forall j\in G_i(A)\backslash N_{\sigma_i},\ \forall i\in\langle n\rangle$. *Then*

$$\sigma(A)\subseteq(\bigcup_{i\in\langle n\rangle}D_i(A))\cup(\bigcup_{\substack{i\in N_{\sigma_i},\,j\in N_{\sigma_j}\\ \{i,j\}\in E_\sigma}}D_{ij}(A)),\tag{1}$$

where

$$D_i(A)=\{z\geq 0\,:\,|z-a_i|\leq S^{(i)}_{N_{\sigma_i}}(A)\},\qquad \forall i\in\langle n\rangle,$$

for all $i\neq j$,

$$D_{ij}(A)=\{z\geq 0\,:\,(|z-a_i|-S^{(i)}_{N_{\sigma_i}}(A))(|z-a_j|-S^{(j)}_{N_{\sigma_j}}(A))\leq \bar{S}^{(i)}_{N_{\sigma_i}}(A)\bar{S}^{(j)}_{N_{\sigma_j}}(A)\}.$$

Proof. For $\forall\sigma\in\sigma(A)$, there are two nonzero vectors $x=(x_1,\cdots,x_n)^T$ and $y=(y_1,\cdots,y_n)^T$ such that

$$\sigma x=Ay,\qquad \sigma y=A^*x.\tag{2}$$

We denote $z_i=\max\{|x_i|,\,|y_i|\},\forall i\in\langle n\rangle,\ z_p=\max\limits_{j\in\langle n\rangle}\{z_j\},\,p\in N_{\sigma_p}$. Without loss of generality, we assume that $z_p=|y_p|\geq|x_p|$. Then the $p-$th equality in (2) implies

$$\sigma x_p-a_{pp}y_p=\sum_{j\in\Gamma^+_p(A)\cap N_{\sigma_p}}a_{pj}y_j+\sum_{j\in\Gamma^+_p(A)\backslash N_{\sigma_p}}a_{pj}y_j\tag{3}$$

$$\sigma y_p-\bar{a}_{pp}x_p=\sum_{j\in\Gamma^-_p(A)\cap N_{\sigma_p}}\bar{a}_{jp}x_j+\sum_{j\in\Gamma^-_p(A)\backslash N_{\sigma_p}}\bar{a}_{jp}x_j.\tag{4}$$

Write $\eta=x_p/y_p$. If $G_p(A)\subseteq N_{\sigma_p}$ or $z_j=0,\quad\forall j\in G_p(A)\backslash N_{\sigma_p}$, then (3) and (4) imply

$$|\sigma\eta-a_{pp}|\leq r^{(p)}_{N_{\sigma_p}}(A)\tag{5}$$

and

$$|\sigma-\eta\bar{a}_{pp}|\leq c^{(p)}_{N_{\sigma_p}}(A),\tag{6}$$

respectively . That $|\eta|\leq 1$. So, if $\sigma\leq a_p$, then $|\sigma-a_p|\leq|\eta||\sigma-a_p|\leq|\sigma\eta-a_{pp}|$, and if $\sigma\geq a_p$ then $|\sigma-a_p|\leq|\sigma-|\eta|a_p|\leq|\sigma-\eta\bar{a}_{pp}|$. Therefore, from (5) and (6) it can be deduced that

$$|\sigma-a_p|\leq S^{(p)}_{N_{\sigma_p}}(A),$$

i.e., $\sigma\in D_p(A)$.

If $\sigma\notin\bigcup\limits_{i\in\langle n\rangle}D_i(A)$, by the above discussions we have $G_p(A)\backslash N_{\sigma_p}\neq\phi$ and $z_q=\max\limits_{j\in G_p(A)\backslash N_{\sigma_p}}\{z_j\}>0$ (otherwise (5) and (6) imply $\sigma\in D_p(A)$). Thus equalities (3) and (4) imply

$$|\sigma-a_p|\leq S^{(p)}_{N_{\sigma_p}}(A)+\bar{S}^{(p)}_{N_{\sigma_p}}(A)\frac{z_q}{z_p}.\tag{7}$$

For $q\in N_{\sigma_q}\subset G_p(A)\backslash N_{\sigma_p}$, we have $G_q(A)\neq\phi$ (otherwise we can deduce $\sigma=a_q$, that is, $\sigma\in D_q(A)$). Similarly, if $z_q=|y_q|\geq|x_q|$,then it is easy to derive the following formula from the $q-$th equality in (2):

$$|\sigma-a_q|\leq S^{(q)}_{N_{\sigma_q}}(A)+\bar{S}^{(q)}_{N_{\sigma_q}}(A)\frac{z_p}{z_q}.\tag{8}$$

Note that $\sigma\notin\bigcup\limits_{i\in\langle n\rangle}D_i(A)$, we have $|\sigma-a_p|>S^{(p)}_{N_{\sigma_p}}(A)$ and $|\sigma-a_q|>S^{(q)}_{N_{\sigma_q}}(A)$. Thus, from (7) and (8) we get

$$(|\sigma-a_p|-S^{(p)}_{N_{\sigma_p}}(A))(|\sigma-a_q|-S^{(q)}_{N_{\sigma_q}}(A))\leq\bar{S}^{(p)}_{N_{\sigma_p}}(A)\bar{S}^{(q)}_{N_{\sigma_q}}(A).\tag{9}$$

Since $q\in G_p(A)\backslash N_{\sigma_p}\neq\phi$, it holds that $\{p,q\}\in E_\sigma$.