Journal of Computational Mathematics, Vol.23, No.2, 2005, 205–210.

ON THE EXTREMAL PROPERTIES OF OPEN COMPOSITE TRAPEZOIDAL FORMULAE *1)

Xing-hua Wang
(*Department of Mathematics, Zhejiang University, Hangzhou 310028, China*)
(*College of Mathematics & Information Science, Wenzhou University, Wenzhou 325035, China*)
Wan Ma
(*College of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, China*)
Xiang-jiang Mi
(*Department of Mathematics, Zhejiang University, Hangzhou 310028, China*)

Abstract

It is found that the open composite trapezoidal formulae are the best quadrature formulae under three different senses.

Mathematics subject classification: 41A55, 65D30.
Key words: Open composite trapezoidal formula, Hölder classes, Sobolev classes, Best quadrature, Exact estimate, Iyengar inequality, Lagrange information.

1. Introduction

Recently, Guessab and Schmeisser, under various different conditions, discussed the "sharp" estimates of the remainder $E(f;x_1)$ for two-symmetric-nodes trapezoidal quadrature formulae with a variable node $x_1 \in [a, \frac{1}{2}(a+b)]$,

$$\int_a^b f(t)\mathrm{d}t = \frac{1}{2}(f(x_1) + f(a+b-x_1))(b-a) + E(f;x_1).$$

Actually, there are many differences among the real meanings of all these so called "sharp" estimates, but they are not distinguished in [1] and thus do not lead to further results[2]. More importantly, restrictions on the numbers and symmetry of nodes of the quadrature formula family make the applications of these formulae greatly inconvenient. Imagine that if in a practical situation the nodes are fixed and hence cannot be chosen freely then we cannot make do on using the above formulae. And even if the nodes may be chosen freely, using the composite version of the above formulae will lead to unnecessary increase of numbers of computing functional values of the integrand. It is not worth doing that in view of computational complexity.

Taking the above reasons into account, we will discuss in this paper the following open composite trapezoidal formulae

$$\int_a^b f(t)\mathrm{d}t \approx Q_{\mathbf{x},\mathrm{OCT}}(f) := f(x_1)(x_1-a) + \sum_{i=1}^{n-1} \frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i) + f(x_n)(b-x_n)$$

for the general set of nodes $\mathbf{x} = (x_1, x_2, \cdots, x_n)$,

$$a \le x_1 < x_2 < \cdots < x_n \le b.$$

* Received September 2, 2003.
1)The work is supported in part by the Special Funds for Major State Basic Research Projects (Grant No. G19990328), and the National and Zhejiang Provincial Natural Science Foundation of China (Grant No. 10471128 and Grant No. 101027).

It goes without saying that the " open " formulae we call here include the "closed" formulae as their special cases of the extreme case allowing $x_1 = a,\ x_n = b$. Obviously, two- symmetric-nodes trapezoidal formulae are only the simplest special cases of the open composite trapezoidal formulae when $\mathbf{x} = (x_1, a+b-x_1)$. Firstly, we must distinguish the definite meanings of various "sharp" estimates. Under the explanation with the definite meaning, the "sharp" estimate should be properly called " exact" estimate. For example, in 1938, Iyengar[3] proved the following inequality when $|f'(t)| \le K$

$$\Big|\int_a^b f(t)\mathrm{d}t - \frac{f(a)+f(b)}{2}(b-a)\Big| \le \frac{K}{4}(b-a)^2 - \frac{(f(b)-f(a))^2}{4K}.$$

Many researchers believe it is a very "good" inequality (see e.g., [4] Sect. 3.7.24), but where on earth is it good? Various generalizations about it still continue to appear (see e.g., [1, 5-9]). But all of them just formally generalize the inequality since a clear understanding of the very essential of the trapezoidal rule indicated by it is not achieved.

In this paper we will consider the exact estimates of the remainder of the open composite trapezoidal formulae $Q_{\mathbf{x},\mathrm{OCT}}(f)$ under three different conditions. It is a surprise to find that the formulae $Q_{\mathbf{x},\mathrm{OCT}}(f)$ themselves have three different extremal properties. From the results of the Nikolskii[10] type estimate for the Hölder classes, we find that $Q_{\mathbf{x},\mathrm{OCT}}(f)$ are the best quadrature formulae in the sense of Sard[11]. From the results of the exact estimates for the first order and a special second order Sobolev class when the Lagrange information is given, we find that $Q_{\mathbf{x},\mathrm{OCT}}(f)$ are also the best quadrature formulae in the sense of Chebyshev for the above two function classes, respectively, i.e., it is a central algorithm for the integral(see [12] for the central algorithm).

2. Main Results and their Proofs

2.1 The best quadrature formulae for the Hölder classes in the sense of Sard

For $K > 0,\quad 0 < \alpha \le 1$, let $KH^\alpha[a,b]$ be the Hölder classes on the interval [a, b], whose definitions are as follows,

$$KH^\alpha[a,b] = \Big\{f:\ |f(t') - f(t)| \le K|t'-t|^\alpha, \quad \forall\ t,\ t' \in [a,b]\Big\}.$$

Theorem 2.1. *Let $f \in KH^\alpha[a,b]$. Then*

$$\Big|\int_a^b f(t)\,dt - Q_{\mathbf{x},\mathrm{OCT}}(f)\Big| \le \frac{K}{\alpha+1}(x_1-a)^{\alpha+1} + \frac{2K}{\alpha+1}\sum_{i=1}^{n-1}\Big(\frac{x_{i+1}-x_i}{2}\Big)^{\alpha+1} + \frac{K}{\alpha+1}(b-x_n)^{\alpha+1}.$$

Moreover, the estimate given by the above inequality is exact in the following sense that there exists a function $f_* \in KH^\alpha[a,b]$ such that the following equality holds

$$\Big|\int_a^b f_*(t)\,dt - Q_{\mathbf{x},\mathbf{w}}(f_*)\Big| = \frac{K}{\alpha+1}(x_1-a)^{\alpha+1} + \frac{2K}{\alpha+1}\sum_{i=1}^{n-1}\Big(\frac{x_{i+1}-x_i}{2}\Big)^{\alpha+1} + \frac{K}{\alpha+1}(b-x_n)^{\alpha+1},$$

where $Q_{\mathbf{x},\mathbf{w}}(f)$ are arbitrary linear quadrature formulae based on the set of nodes $\mathbf{x}$, i. e.

$$Q_{\mathbf{x},\mathbf{w}}(f) := \sum_{i=1}^n w_i f(x_i).$$

Proof. We have

$$\int_a^b f(t)\mathrm{d}t - Q_{\mathbf{x},\mathrm{OCT}}(f) = \int_a^{x_1}(f(t)-f(x_1))\mathrm{d}t + \sum_{i=1}^{n-1}\Big\{\int_{x_i}^{c_i}(f(t)-f(x_i))\mathrm{d}t + \int_{c_i}^{x_{i+1}}(f(t)-f(x_{i+1}))\mathrm{d}t\Big\} + \int_{x_n}^b (f(t)-f(x_n))\mathrm{d}t,$$