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SMOOTHING BY CONVEX QUADRATIC PROGRAMMING *1)

Bing-sheng He Yu-mei Wang
(*Department of Mathematics, Nanjing University, Nanjing 210093, China*)

Abstract

In this paper, we study the relaxed smoothing problems with general closed convex constraints. It is pointed out that such problems can be converted to a convex quadratic minimization problem for which there are good programs in software libraries.

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1. Introduction

Let

$$x_1 < x_2 < \cdots < x_n < x_{n+1}$$

and

$$y_1, y_2, \ldots, y_n, y_{n+1} = y_1.$$

The mathematical form of the problems considered in this paper is to find a twice continuous differentiable periodic function $g(x)$ with $g(x_{n+i}) = g(x_i)$, such that $g(x)$ is the optimal solution of the following problem:

$$\min \quad \int_{x_1}^{x_{n+1}} |g''(x)|^2 dx \tag{1.1}$$

$$\text{s. t} \quad u \in \Omega \tag{1.2}$$

where

$$u = (u_1, u_2, \ldots, u_n)^T, \qquad u_i = \frac{g(x_i) - y_i}{\delta y_i}, \tag{1.3}$$

$\delta y_i, i = 1, \ldots, n$ are given positive numbers and Ω is a closed convex set in R^n. We refer the problem to *relaxed smoothing problem* whenever $\Omega \neq \{0\}$. For $\Omega = \{v \in R^n \mid \|v\|_2 \leq r\}$, the problem was investigated by Reinsch [2] and it was converted to a smooth convex unconstrained optimization. Problem (1.1) with general closed convex constraints have more applications, for example, $\Omega = \{v \in R^n \mid \|v\|_\infty \leq r\}$ is also interesting in real problems.

It is well known that the solution of the non-relaxed problem of (1.1) is a spline function. We will prove that the solution of the relaxed smoothing problem with general closed convex constraints is the spline function $g(x) \in C^2$ of the following form:

$$g(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, \quad x \in [x_i, x_{i+1}). \tag{1.4}$$

Then the task of solving problem (1.1)-(1.2) is to find $a_i, b_i, c_i, d_i, i = 1, \ldots, n$.

In next section, we summarize some notations and the basic relations of the spline function. Section 3 illustrates that the coefficients of the spline function can be obtained by solving a

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convex quadratic programming. Finally, in Section 4, we prove that the obtained spline function is the solution of the original problem and give our conclusions.

2. Notations and the Basic Relations

For analysis convenience, we need the following notations. Let $h_i := x_{i+1} - x_i$,

$$D = \begin{pmatrix} \delta y_1 & & & \\ & \delta y_2 & & \\ & & \ddots & \\ & & & \delta y_n \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix}$$

be diagonal matrices in $R^{n\times n}$. Denote

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}.$$

Note that a, b, c, d are unknown vectors. Since $g(x_i) = a_i$, using these notations, the relation (1.3) can be written as

$$u = D^{-1}(a - y). \tag{2.1}$$

In addition, we needs the following permutation matrix

$$P := \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}.$$

For this matrix P we have $P^T P = I$,

$$Pa = \begin{pmatrix} a_2 \\ \vdots \\ a_n \\ a_1 \end{pmatrix} \qquad \text{and} \qquad P^T a = \begin{pmatrix} a_n \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}.$$

Now, let us list the basic properties of the periodic spline function $g(x) \in C^2$. First, since $g(x_{i+1}^-) = g(x_{i+1}^+)$, we have $a_i + b_i h_i + c_i h_i^2 + d_i h_i^3 = a_{i+1}$ and thus

$$a + Hb + H^2 c + H^3 d = Pa. \tag{2.2}$$

In addition, because $g'(x_{i+1}^-) = g'(x_{i+1}^+)$, we have $b_i + 2c_i h_i + 3d_i h_i^2 = b_{i+1}$ and

$$b + 2Hc + 3H^2 d = Pb. \tag{2.3}$$

Finally, since $g''(x_{i+1}^-) = g''(x_{i+1}^+)$, we have $c_i + 3d_i h_i = c_{i+1}$ and thus

$$c + 3Hd = Pc. \tag{2.4}$$