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THE OPTIMAL ORDER ERROR ESTIMATES FOR FINITE ELEMENT APPROXIMATIONS TO HYPERBOLIC PROBLEMS*1)

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Abstract

In this paper, the linear finite element approximation to the positive and symmetric, linear hyperbolic systems is analyzed and an $O(h^2)$ order error estimate is established under the conditions of strongly regular triangulation and the H^3-regularity for the exact solutions. The convergence analysis is based on some superclose estimates derived in this paper. Our method and result here are also applicable to general hyperbolic problems. Finally, we discuss the linearized shallow water system of equations.

Mathematics subject classification: 65M, 65N.
Key words: Hyperbolic problems, Finite element approximations, Optimal error estimates.

1. Introduction

Since 1970's, finite element method for solving partial differential equations has been successfully applied to elliptic and parabolic problems, however, it is still not very popular for hyperbolic problems. In view of that compared with the difference method, finite element method is more flexible and adaptive, and easier to mathematically analyze, recently finite element methods for hyperbolic problems have attracted more and more attention; see, e.g., [1-5] for the Galerkin method; [6-13] for the discontinuous Galerkin method; [14-17] for the Petrov-Galerkin method; and [18-21] for the streamline diffusion method.

It is well known that for the k-th order finite element approximations to elliptic or parabolic problems, the optimal order error estimate in L_2 norm is of $O(h^{k+1})$ order with the exact solution u in $H^{k+1}(\Omega)$. However, for linear hyperbolic problems, it is still an unsolved completely problem that whether or not the finite element solutions admit this optimal order estimate. Generally speaking, the convergence order of Galerkin method for hyperbolic problems is of $O(h^k)$ order, that is one order lower than the approximation order of finite element space; cf. [1] and [2]. And in [1], Dupont gave a counterexample by using third order Hermit element to indicate that this convergence order is sharp. Since then, in order to obtain the high accuracy and cope with the lower regularity of hyperbolic problems, the discontinuous Galerkin method is proposed and used extensively in this area; cf. [6],[7],[8],[9],[12] and [13]. By this method, the convergence order can be improved to $O(h^{k+\frac{1}{2}})$, and recently some superconvergence results are also given in [22] for elliptic problem by using discontinuous Galerkin method.

In the context of Galerkin method, under some assumptions on the finite element partition and regularity of the exact solution, it is possible to obtain the optimal order error estimates when linear finite elements are used; see, e.g.,[3] for bilinear rectangular element; and [5] for linear triangular element imposed on uniform mesh partition. Obviously, the condition of uniform mesh partition is not very interesting in the practical case.

In this paper, we will discuss the linear finite element approximation to positive and symmetric hyperbolic systems. Under the conditions of strongly regular triangulation (cf. [23]) and

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H^3-regularity for the exact solutions, the optimal order error estimates are established. The theoretical tools for the error analysis are some superclose estimates that are also derived in this paper. Our method and result here are also applicable to general hyperbolic problems. To author's knowledge, very few optimal convergence order can be reached for hyperbolic problems, even in one dimensional case. Hence, our research work in this paper is theoretically significant.

Let $\Omega \subset R^2$ be a polygonal domain, $J_h = \{e\}$ be the finite element partition of domain Ω parameterized by mesh size h so that $\overline{\Omega} = \cup_{e\in J_h}\{\,\overline{e}\,\}$. Introduce the linear finite element space S_h defined by

$$S_h = \{\, v \in C(\overline{\Omega}) \bigcap H^1(\Omega) : v|_e \text{ is linear, } \forall e \in J_h \,\}.$$

We will use the standard notation for the Sobolev spaces $W_p^m(\Omega)$ with corresponding norms and seminorms, and when $p = 2$, $W_2^m(\Omega) = H^m(\Omega)$, $\|\cdot\|_{m,2} = \|\cdot\|_m$. Denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the standard inner product and norm in $L_2(\Omega)$ space. Let X be a Banach space, constant $T > 0$, we will also use the space,

$$L_p(0,T;X) = \{\, v(t) : (0,T) \to X : \ \|v\|_{L_p(X)} = \Big(\int_0^T \|v(t)\|_X^p \, dt\,\Big)^{\frac{1}{p}} < \infty \,\}.$$

In this paper, letter C represents a generic constant independent of mesh size h.

The plan of this paper is as follows. In section 2, some superclose estimates for interpolation are established. In section 3, the linear finite element approximations are analyzed for steady and nonsteady positive and symmetric hyperbolic systems, respectively, and the optimal order error estimates are derived. Finally, we will discuss the linearized shallow water system of equations.

2. Superclose Estimates

Definition 2.1. *Let $e = \triangle p_1p_2p_3$, $e' = \triangle p_1'p_2'p_3'$, and e and e' be two adjacent triangle elements sharing a common edge in J_h. The quadrilateral $\overline{e}\cup\overline{e'}$ is called as an approximate parallelogram if (see figure 1)*

$$|\,\overrightarrow{p_1p_2} + \overrightarrow{p_1'p_2'}\,| = O(h^2)\,, \qquad |\,\overrightarrow{p_2p_3} + \overrightarrow{p_2'p_3'}\,| = O(h^2). \tag{2.1}$$

Definition 2.2. *A triangulation J_h is called as strongly regular, if any two adjacent triangular elements in J_h form an approximate parallelogram (see figure 1).*

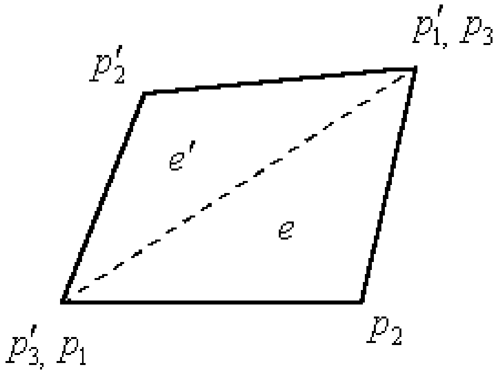

Figure 1. approximating parallelogram