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ON THE ERROR ESTIMATES FOR THE ROTATIONAL PRESSURE-CORRECTION PROJECTION SPECTRAL METHODS FOR THE UNSTEADY STOKES EQUATIONS *1)

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Abstract

This paper provides an analysis of the rotational form of the pressure-correction methods by spectral approximations for the unsteady Stokes equations. Error estimates in finite time for the fully discrete case are given. Numerical experiences using both spectral and spectral element methods are carried out to confirm the theoretical results.

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Key words: Stokes equations, Projection methods, Spectral methods.

1. Introduction

Efficient solution of the Stokes equations is dependent upon the availability of fast solvers for the pressure operator, as the pressure characteristic propagation speed is infinite for unsteady incompressible flow. Generally, there are two principal ways to discretize the unsteady Stokes equations in time. One way is to keep the velocity and the pressure coupled, and at each time step it needs to solve the generalized Stokes problem which is the most computationally expensive. A common technique for solving the algebraic system, stemming from discretization of the Stokes equations, is the Uzawa algorithm. An Uzawa algorithm uses block Gaussian elimination and back substitution for the pressure and the velocity yielding two positive definite symmetric systems (see e.g. [19] and the references therein). This decoupling procedure has been proven to be attractive than a direct algorithm. However the classical Uzawa algorithm suffers from expensive solve of the pressure system as the pressure matrix involves the inverses of the Helmholtz systems. This disadvantage could be overcome by using an additional splitting technique. This approach has a common foundation with traditional splitting approaches which leads to a Poisson equation for the pressure except that, in the former case, the splitting is effected in the discrete form of the equations. Such an approach was analyzed and applied to the various computations in the papers of Perot [21], Couzy *et al.* [9] and Fischer [11], but no rigorous error estimate is available. The disadvantage of the Uzawa-based algorithm is that a discrete form of the Ladyshenskaya-Brezzi-Babuška condition(LBB condition, [5]) must be satisfied for obtaining the unique discrete solution. This means that for a high-order spectral approximation, the degree of approximation for the pressure must be taken two degrees lower than that for the velocity [20]. It is the so called $P_N \times P_{N-2}$ method. There exist some methods that make use other space pairs than $P_N \times P_{N-2}$, we refer to [6] for detailed description of these methods.

Another way to discretize the continuous unsteady Stokes equations is provided by the class of projection methods. This class of approaches has been introduced by Chorin [7, 8] and Temam [28]. They are based on a particular time-discretization of the equations governing

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viscous incompressible flows, in which the viscosity and the incompressibility of the fluid are dealt within two separate steps. By doing that, the original problem is reformulated into two new and simpler problems. The theory of saddle-point problems is then no longer needed, that is to say, the LBB condition is not needed; as a consequence, the degree of approximation for the velocity and the pressure can be taken the same, yielding a simpler-to-implement numerical scheme.

The projection algorithm can be interpreted as a predictor-corrector strategy, which can be essentially classed into two families: classical fractional step methods and pressure-correction methods. The classical fractional step methods have only first order convergence rate due to the fact that it is basically an artificial compressibility technique [24, 25]. Different choices of the pressure boundary condition have been discussed to improve the efficiency of this kind of methods (see [17] for instance). The pressure-correction methods consist of two time substeps: first we make the pressure explicit in the convection-diffusion step, and then compute its increment (correction) in the projection step. Second-order Error estimates in the L^2-norm for the velocity have been proved in the several papers [10, 26, 15, 27] for different cases. However the pressure accuracy in a standard pressure-correction scheme can be at most of first-order in the L^2-norm, as shown by Strikwerda and Lee in [27]. In 1996, Timmermans, Minnev and Van De Vosse introduced in [30] a modified pressure-correction scheme. They analyzed this approach by means of an analytical test solution in the case of spectral element spatial discretization, and showed that the L^2 errors of both the velocity and the gradient of the pressure are of second-order, but the computed maximum pressure failed to converge due to the presence of the corners. Recently, Guermond and Shen [14] reviewed this modified version of the pressure-correction schemes. They termed it as the *rotational form* of the pressure-correction schemes, and showed that the pressure approximation in the L^2-norm is indeed $\frac{3}{2}$-order accurate. A detail proof was given in the semi-discrete case.

The main task of the present paper is to provide a rigorous stability and error analysis for the rotational form of the pressure-correction schemes in the fully discrete case using a Galerkin spectral approximation. In order to get the optimal error estimates, we still assume that the approximate velocity and pressure space pair satisfies the LBB condition. We prove that the velocity error in time and in space is $O(\delta t^2 + N^{-m})$ for the $l^2(L^2(\Omega)^2)$-norm and that the pressure error is $O(\delta t^{\frac{3}{2}} + N^{-m})$ for the $l^2(L^2(\Omega))$-norm, where N is the polynomial degree used to approximate the velocity, δt is the time step, m is the regularity of the exact pressure solution. Our numerical experiences are in good agreement with the above theoretical results for the velocity, but the computed pressure appears to have higher order accuracy in time than $O(\delta t^{\frac{3}{2}})$. Particularly our numerical results show that the pressure accuracy is sensible to the kinematics viscosity in the case of singular computational domain. In the case of square domain, for the $L^2(\Omega)$ norm at a given time $T \geq 1$, the pressure accuracy is less than 2-order. But in the case of smooth domain, the pressure accuracy is fully 2-order. This conforms to the results given in [30] and [14].

We should emphasize that, in order to gain maximal simplicity in the implementation, our numerical experiences use spectral and spectral element approximations of $P_N \times P_N$ version (we refer to [14] for the $P_N \times P_{N-2}$ version). As already indicated in [14], there are larger pressure errors at the domain corners for the projection $P_N \times P_{N-2}$ spectral methods. Our numerical results show that these larger pressure errors will be further enlarged if the $P_N \times P_N$ version is used, and the maximum pressure error fails to converge due to the presence of the corners, specially for very small δt. However we will show that this failure can be efficiently overcome by a simple filtering procedure, which consists in projecting the computed pressure into P_{N-2} space at each time step. This procedure is, in some sense, equivalent to the $P_N \times P_{N-2}$ version, but is easier to implement. We refer it to as the filtered $P_N \times P_N$ version.

The outline of this paper is as follow: in Section 2 we recall the basic steps of projection-type methods, and define their spectral approximation formulations. In section 3 we provide rigorous