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ON THE MINIMAL NONNEGATIVE SOLUTION OF NONSYMMETRIC ALGEBRAIC RICCATI EQUATION [*1)]

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Abstract

We study perturbation bound and structured condition number about the minimal nonnegative solution of nonsymmetric algebraic Riccati equation, obtaining a sharp perturbation bound and an accurate condition number. By using the matrix sign function method we present a new method for finding the minimal nonnegative solution of this algebraic Riccati equation. Based on this new method, we show how to compute the desired M-matrix solution of the quadratic matrix equation $X^2 - EX - F = 0$ by connecting it with the nonsymmetric algebraic Riccati equation, where E is a diagonal matrix and F is an M-matrix.

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Key words: Nonsymmetric algebraic Riccati equation, Minimal nonnegative solution, Matrix sign function, Quadratic matrix equation.

1. Introduction

In this paper, we will mainly study the nonsymmetric *algebraic Riccati equation* (ARE)

$$XCX - XD - AX + B = 0, \tag{1}$$

where A, B, C, D are given real matrices of sizes $m \times m$, $m \times n$, $n \times m$ and $n \times n$, respectively. To this end, let us define two $(m+n) \times (m+n)$ matrices H and K as follows:

$$H = \begin{pmatrix} D & C \\ -B & -A \end{pmatrix}, \qquad K = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}. \tag{2}$$

We will focus on the exploration of the minimal nonnegative solution of the ARE(1) by making use of the invariant subspace of the matrix H when K is a nonsingular M-matrix.

We have noticed that sensitivity analysis about other types of algebraic Riccati equations were studied in depth in [17, 18, 10, 6], and direct methods about the linear matrix equations, the special cases of the algebraic Riccati equations, were presented in detail in [8, 9].

This paper is organized as follows. After reviewing some basic notations and results associated with the nonsymmetric ARE(1) in section 2, we give a perturbation bound for the minimal nonnegative solution of the ARE(1) in section 3. A structured condition number is derived mathematically and verified numerically in section 4. Then, we present a matrix sign function method for finding the minimal nonnegative solution in section 5; this method can also be used to find the desired M-matrix solution of the quadratic matrix equation $X^2 - EX - F = 0$, with E a diagonal matrix and F an M-matrix. Finally, in section 6 we use some numerical examples to illustrate the correctness of our theory and the feasibility of our methods.

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2. Basic Notations and Results

Given two matrices $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m\times n}$, we write $A \geq B$ ($A > B$) if $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$) hold for all i and j, and we call the matrix A positive (nonnegative), if $A > 0$ ($A \geq 0$).

Let $A \in \mathbb{R}^{n\times n}$. It is called a Z-matrix if all of its off-diagonal elements are nonpositive. Clearly, a Z-matrix $A \in \mathbb{R}^{n\times n}$ can be represented as $A = sI - B$, with $B \geq 0$. In particular, when $s > \rho(B)$, the spectral radius of the matrix B, A turns to a nonsingular M-matrix, and when $s = \rho(B)$, it turns to a singular M-matrix. We use $\lambda(A)$ to denote the spectrum of the matrix A, $\sigma_{\min}(A)$ the smallest singular value of A, and $\mathcal{R}(A)$ the range space spanned by the columns of the matrix A.

The open left (right) half plane is denoted by $\mathbb{C}_<$ ($\mathbb{C}_>$), and the closed left (right) half plane is denoted by $\mathbb{C}_\leq$ ($\mathbb{C}_\geq$), respectively. In addition, we use $\|\cdot\|$ to denote any consistent matrix norm on $\mathbb{C}^{n\times n}$ unless it is claimed explicitly. In particular, we use $\|\cdot\|_2$ and $\|\cdot\|_F$ to denote the spectral and the Frobenius norms of a matrix, respectively.

We recall that the separation of two matrices $B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times m}$ can be defined as follows. See [14].

$$\mathrm{sep}(B, C) := \inf\{\|PB - CP\| \mid B \in \mathbb{R}^{n\times n},\ C \in \mathbb{R}^{m\times m} \text{ and } P \in \mathbb{R}^{m\times n},\ \text{with } \|P\| = 1\}. \quad (3)$$

When the norm in (3) is specified to be the Frobenius norm, we denote the separation $\mathrm{sep}(B, C)$ by $\mathrm{sep}_F(B, C)$.

The following properties about an M-matrix can be found in [1].

Lemma 2.1. *[1] Given a Z-matrix $A \in \mathbb{R}^{n\times n}$. Then the following statements are equivalent:*

(a) A is a nonsingular M-matrix;

(b) $A^{-1} \geq 0$;

(c) $Av > 0$ holds for some vector $v > 0$;

(d) $\lambda(A) \subset \mathbb{C}_>$.

For the nonsymmetric ARE(1), from [2, 3] we know that the following results hold.

Lemma 2.2. *If the matrix K defined in (2) is a nonsingular M-matrix, then the ARE(1) has a minimal nonnegative solution S that satisfies that both matrices $D_C := D - CS$ and $A_C := A - SC$ are nonsingular M-matrices.*

Lemma 2.3. *If the matrix K defined in (2) is a nonsingular M-matrix, then the matrix H defined in (2) has n eigenvalues in $\mathbb{C}_>$ and m eigenvalues in $\mathbb{C}_<$.*

Lemma 2.4. *If the matrix K defined in (2) is a nonsingular M-matrix and S is a minimal nonnegative solution of the ARE(1), then*

$$\begin{pmatrix} I & 0 \\ S & I \end{pmatrix} \begin{pmatrix} D & C \\ -B & -A \end{pmatrix} \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix} = \begin{pmatrix} D - CS & C \\ 0 & -(A - SC) \end{pmatrix}.$$

It then follows that the column space of the matrix

$$\begin{pmatrix} I \\ -S \end{pmatrix}$$

is the unique invariant subspace of the matrix H associated with its n eigenvalues in $\mathbb{C}_>$.