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# LEAST-SQUARES MIXED FINITE ELEMENT METHODS FOR THE INCOMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS *

Shao-qin Gao
(*ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)
(*Department of Mathemaitcs, Hebei Unviersity, Baoding 071002, China*)

**Abstract**

Least-squares mixed finite element methods are proposed and analyzed for the incompressible magnetohydrodynamic equations, where the two vorticities are additionally introduced as independent variables in order that the primal equations are transformed into the first-order systems. We show that there hold the coerciveness and the optimal error bound in appropriate norms for all variables under consideration, which can be approximated by all kinds of continuous element. Consequently, the Babuška-Brezzi condition (i.e. the inf-sup condition) and the indefiniteness are avoided which are essential features of the classical mixed methods.

*Mathematics subject classification:* 65N30.
*Key words*: The incompressible magnetohydrodynamic equation, Vorticity, Least-squares mixed finite element method.

## 1. Introduction

Many problems involve systems of partial differential equations in several variables, variational problems derived in a standard manner often correspond to saddle-point optimization problem. It is now well understood that the finite element spaces approximating different physical quantities can not be chosen independently and have to satisfy the inf-sup condition.

In recent years there has been significant interest in least-squares methods, considered as an alternative to the saddle point formulations and circumventing the inf-sup condition. Many studies have already be devoted to the least-squares method, for theoretical and numerical results, let us just mention those by Z.Cai, T.Manteuffel and S.McCormick [4, 5, 6], A.I.Pehlivanov, G.F.Carey and R.D.Lazarov [15], Dan-Ping Yang [19, 20], B.N.Jiang [12,13] and Huo-yuan Duan [9]. The stationary incompressible magnetohydrodynamics(MHD) we consider here results from a coupling between the stationary incompressible Navier-Stokes equations and the stationary Maxwell equations. It governs the behavior of an incompressible fluid carrying an electrical current in presence of a magnetic field. We study the linear MHD equations (cf. [10]) using the least-squares mixed finite element method.

The paper is organized as follows: In section 2, we introduce some notations, Hilbert spaces and inequalities. Section 3 is concerned with the presentation of the equations and the derivation of the least-squares formulation of the linear MHD equations and its coerciveness. In section 4, we give the finite element approximation and obtain the basic error bounds.

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## 2. The Preparations

First we recall some notations. Let $\Omega \subset R^3$ is an open bounded domain with boundary $\Gamma = \partial\Omega$, $\mathbf{n}$ is unit normal vector to $\Gamma$. We introduce the following Sobolev spaces

$$\begin{aligned}
L^2(\Omega) &= \{v; \int_\Omega v^2 < \infty\}, \\
L_0^2(\Omega) &= \{q \in L^2(\Omega); \int_\Omega q = 0\}, \\
H^m(\Omega) &= \{\partial^\gamma v \in L^2(\Omega),\ 0 \le |\gamma| \le m\},\ (m \ge 1), \\
H_0^1(\Omega) &= \{v \in L^2(\Omega);\ \nabla v \in (L^2(\Omega))^3,\ v_{|\Gamma} = 0\}, \\
H(\mathbf{curl};\Omega) &= \{\mathbf{u} \in (L^2(\Omega))^3;\ \mathbf{curl}\ \mathbf{u} \in (L^2(\Omega))^3\}, \\
H(\mathrm{div};\Omega) &= \{\mathbf{u} \in (L^2(\Omega))^3;\ \mathrm{div}\ \mathbf{u} \in L^2(\Omega)\}, \\
H_0(\mathbf{curl};\Omega) &= \{\mathbf{u} \in H(\mathbf{curl};\Omega);\ \mathbf{u} \times \mathbf{n}_{|\Gamma} = \mathbf{0}\}, \\
H_0(\mathrm{div};\Omega) &= \{\mathbf{u} \in H(\mathrm{div};\Omega);\ \mathbf{u} \cdot \mathbf{n}_{|\Gamma} = 0\}.
\end{aligned}$$

Two Green's formulae of integration by parts (cf. [11]), two equalities and one inequality are as follows:

$$\begin{aligned}
(\mathbf{u}, \nabla\phi) + (\mathrm{div}\ \mathbf{u}, \phi) &= < \mathbf{u} \cdot \mathbf{n},\ \phi >_{|\Gamma} \quad \forall\ \mathbf{u} \in H(\mathrm{div};\Omega), \quad \forall\ \phi \in H^1(\Omega). && (1) \\
(\mathbf{curl}\ \mathbf{u}, \mathbf{v}) - (\mathbf{u}, \mathbf{curl}\ \mathbf{v}) &= < \mathbf{u} \times \mathbf{n},\ \mathbf{v} >_\Gamma \quad \forall\ \mathbf{u} \in H(\mathbf{curl};\Omega), \ \forall\ \mathbf{v} \in (H^1(\Omega))^3. && (2) \\
(\mathbf{a} \cdot \nabla\mathbf{u}, \mathbf{u}) + \frac{1}{2}(\mathrm{div}\ \mathbf{a}, |\mathbf{u}|^2) &= 0 \quad \forall\ \mathbf{u} \in (H_0^1(\Omega))^3. && (3) \\
(\mathbf{b} \times \mathbf{curl}\ \mathbf{B}, \mathbf{u}) &= (\mathbf{u} \times \mathbf{b}, \mathbf{curl}\ \mathbf{B})\ \forall\ \mathbf{b}, \mathbf{u} \in (L^2(\Omega))^3, \ \forall\ \mathbf{B} \in H(\mathbf{curl}, \Omega). && (4) \\
||\mathbf{a} \times \mathbf{b}||_0^2 &\le C||\mathbf{a}||_{0,\infty}^2 ||\mathbf{b}||_0^2 \qquad \forall\ \mathbf{a}, \mathbf{b} \in (L^2(\Omega))^3. && (5)
\end{aligned}$$

Where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^3$.

**Proposition 2.1 (cf. [2,11,14,18]).** *Assume that $\Omega \subset R^3$ is a simply-connected and bounded domain with a Lipschitz continuous boundary $\Gamma$. Then*

$$||\mathbf{u}||_0 \le C\{||\mathbf{curl}\ \mathbf{u}||_0 + ||\mathrm{div}\ \mathbf{u}||_0\} \ \ \forall\ \mathbf{u} \in H_0(\mathrm{div};\Omega) \cap H(\mathbf{curl};\Omega). \tag{6}$$

**Proposition 2.2 (cf. [11,8]).** *Assume that $\Omega \subset R^3$ is a simply-connected and bounded domain with $C^{1,1}$ boundary $\Gamma$, or is a bounded and convex polyhedron. Then*

$$|\mathbf{u}|_1 \le C\{||\mathbf{curl}\ \mathbf{u}||_0 + ||\mathrm{div}\ \mathbf{u}||_0\} \ \ \forall\ \mathbf{u} \in H_0(\mathrm{div};\Omega) \cap (H^1(\Omega))^3. \tag{7}$$