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PIECEWISE SEMIALGEBRAIC SETS *1)

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Abstract

Semialgebraic sets are objects which are truly a special feature of real algebraic geometry. This paper presents the piecewise semialgebraic set, which is the subset of R^n satisfying a boolean combination of multivariate spline equations and inequalities with real coefficients. Moreover, the stability under projection and the dimension of C^μ piecewise semialgebraic sets are also discussed.

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1. Introduction

Semialgebraic sets and semialgebraic functions(i.e. functions having semialgebraic graph) are objects which are truly a special feature of real algebraic geometry. This class of sets has remarkable stability properties, of which the most important is stability under projection. Practically all the useful constructions with semialgebraic sets have also a very pleasant topological structure: they have good stratifications. Furthermore, semialgebraic functions grow in a very well controlled way. The recent researches of the semialgebraic sets refer to [1,2,3,4,8]

A piecewise algebraic variety(curve) is the zero set of some multivariate(bivariate) splines. As the generalization of the classical algebraic variety(curve), the piecewise algebraic variety(curve) is not only very important for several practical areas such as CAD(Computer-Aided Design), CAM(Computer-Aided Manufacture), CAE(Computer-Aided Engineering), and image processing, but also a useful tool for studying other subjects[13,14,17]. Wang et al. have done a lot of work concerning the piecewise algebraic varieties and curves[7,9,10,12−22].

A piecewise semialgebraic subset of R^n is the subset of $(x_1, \cdots, x_n)$ in R^n satisfying a boolean combination of spline equations and inequalities with real coefficients. It is the generalization of the semialgebraic set. The piecewise algebraic variety(curve) is the degeneration of the piecewise semialgebraic set.

This paper deals with piecewise semialgebraic sets over real closed field R. First of all, we present the piecewise algebraic varieties and piecewise semialgebraic sets. In section 3, we discuss the stability of the piecewise semialgebraic sets under projection and several applications of this property are also investigated. The dimension of piecewise semialgebraic sets are discussed in section 4.

2. Piecewise Algebraic Varieties and Piecewise Semialgebraic Sets

Let R be a real closed field. Using finite number of hypersurfaces in R^n, we partition R^n into finite number of simply connected regions, which are called the partition cells. Denote by Δ the partition of the region R^n which is the union of all partition cells $\delta_1, \cdots, \delta_T$ and their

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edges $S_1, \cdots, S_E$. $S_1, \cdots, S_E$ are the algebraic hypersurfaces or algebraic families of dimension $\leq n$ which are called the partition net surfaces.

Denote by $P(\Delta)$ the piecewise polynomial ring with respect to partition Δ on R^n as follows

$$P(\Delta) := \{f | f|_{\delta_i} = f_i \in R[x_1, \cdots, x_n], i = 1, 2, \cdots, T\}.$$

For integer $\mu \geq 0$, let

$$S^\mu(\Delta) := \{f | f \in C^\mu(\Delta) \cap P(\Delta)\}.$$

$S^\mu(\Delta)$ is called the C^μ *spline ring*. In fact, it is a Nöther ring[13,14].

Definition 2.1. [10,14] *A subset $X \subset R^n$ is called a C^μ* piecewise algebraic variety *if there exist $f_1, f_2, \cdots, f_r \in S^\mu(\Delta)$, then*

$$X = Z(f_1, f_2, \cdots, f_r) = \{x \in R^n | f_i(x) = 0, i = 1, 2, \cdots, r\}. \tag{1}$$

Theorem 2.1. [10,14] *Let $X \subset R^n$ be a C^μ piecewise algebraic variety. Then the set*

$$I(X) := \{f \in S^\mu(\Delta) | f(x) = 0, \forall x \in X\} \tag{2}$$

yield an ideal of $S^\mu(\Delta)$.

Theorem 2.2. [10,14] *The union of two C^μ piecewise algebraic varieties is a C^μ piecewise algebraic variety. The intersection of many C^μ piecewise algebraic varieties is a C^μ piecewise algebraic variety. The empty set and R^n are C^μ piecewise algebraic varieties.*

Definition 2.2. [10,14] *For any partition Δ of R^n. The topology $\Im^\mu_\Delta = \{R^n \setminus X | X \subset R^n$ is a C^μ piecewise algebraic variety $\}$ is called C^μ* Zariski topology *by taking the closed subsets as the C^μ piecewise algebraic varieties.*

Definition 2.3. [10,14] *Let $X \subset R^n$ be a nonempty C^μ piecewise algebraic variety. If X can be expressed as the union of two nonempty proper closed sets X_1 and X_2, then X is called reducible, otherwise X is called irreducible.*

Theorem 2.3. [10,14]

(1) $\Im^\mu_\Delta$ is a Nöther topology, that is, any irreducible descent closed set sequence $X_1 \supset X_2 \supset \cdots$ is a finite sequence.

(2) Every C^μ piecewise algebraic variety can be expressed as the union of finite number of irreducible C^μ piecewise algebraic varieties.

(3) Let $X \subset R^n$ be a C^μ piecewise algebraic variety. Then X is irreducible if and only if $I(X)$ is a prime ideal of $S^\mu(\Delta)$.

(4) A maximal ideal of $S^\mu(\Delta)$ corresponds to a minimal irreducible closed subset of C^μ piecewise algebraic varieties.

Definition 2.4. *A subset of R^n satisfying*

$$\cup_i \cap_j \{x \in R^n | f_{ij}(x) *_{ij} \quad 0\}$$

is called a C^μ piecewise semialgebraic set, *where $f_{ij} \in S^\mu(\Delta)$ and the $*_{ij}$ are either $=$ or $\neq$ or $>$ or $\geq$.*