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ON HERMITIAN POSITIVE DEFINITE SOLUTION OF NONLINEAR MATRIX EQUATION $X + A^*X^{-2}A = Q$ *1)

Xiao-xia Guo
(*ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

Based on the fixed-point theory, we study the existence and the uniqueness of the maximal Hermitian positive definite solution of the nonlinear matrix equation $X + A^*X^{-2}A = Q$, where Q is a square Hermitian positive definite matrix and A^* is the conjugate transpose of the matrix A. We also demonstrate some essential properties and analyze the sensitivity of this solution. In addition, we derive computable error bounds about the approximations to the maximal Hermitian positive definite solution of the nonlinear matrix equation $X + A^*X^{-2}A = Q$. At last, we further generalize these results to the nonlinear matrix equation $X + A^*X^{-n}A = Q$, where $n \geq 2$ is a given positive integer.

Mathematics subject classification: 65F10, 65F15, 65N30.
Key words: Nonlinear matrix equation, Hermitian positive definite solution, Sensitivity analysis, Error bound.

1. Introduction

Consider the *nonlinear matrix equation* **(NME)**

$$X + A^*X^{-2}A = Q, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a nonsingular matrix, and $Q \in \mathbb{C}^{n\times n}$ is a *Hermitian positive definite* **(HPD)** matrix. Here, we use $\mathbb{C}^{n\times n}$ to denote the set of all $n\times n$ complex matrices, and A^* the conjugate transpose of the matrix A.

Nonlinear matrix equations of type (1.1) often arise in dynamic programming, stochastic filtering, control theory, statistics, and so on. See [9]. It can be categorized into a general system of nonlinear equations in the $\mathbb{C}^{n^2}$ space (see [25, 4, 6, 7]), which includes the linear and nonlinear matrix equations recently discussed in [1, 18, 19, 20, 21, 14, 8] as special cases.

The nonlinear matrix equations $X \pm A^*X^{-1}A = Q$ have been extensively studied by several authors[9, 13, 10, 11, 24, 26], and some properties of their solutions have been obtained. In addition, Xu[23] has given the perturbation analysis of the maximal solution of the nonlinear matrix equation $X + A^*X^{-1}A = Q$. When $Q = I$, the identity matrix, many authors have studied the properties of the (Hermitian) positive definite solutions of the NME(1.1). See [16, 17, 27, 22, 15] for details. However, when Q is a general HPD matrix, the NME(1.1) becomes more complicated and little is known yet about properties of its solutions.

In this paper, we discuss existence and uniqueness of the HPD solutions of the NME(1.1). Moreover, we reveal essential properties of these HPD solutions and their maximal one. In particular, for the maximal HPD solution of the NME(1.1), we investigate its sensitivity property in detail and, based upon this, we derive a computable bound for its numerical approximations. Besides, we further generalize all of these results to the nonlinear matrix equation $X + A^*X^{-n}A = Q$, where $n \geq 2$ is a given positive integer.

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The following notations are used throughout this paper. For $A \in \mathbb{C}^{n\times n}$, we use $\lambda(A)$ to denote its eigenvalue set and $\|A\|$ its spectral norm, i.e.,

$$\lambda(A) = \{\lambda \mid \lambda \text{ is an eigenvalue of the matrix } A\}$$

and

$$\|A\| = \sqrt{\max_i \lambda_i(A^*A)},$$

where $\lambda_i(\cdot)$ is the i-th eigenvalue of the corresponding matrix. In particular, we use $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ to represent the minimal and the maximal eigenvalues of a Hermitian matrix, respectively. For $A, B \in \mathbb{C}^{n\times n}$, we write $A \succ B$ ($A \succeq B$) if both A and B are Hermitian and $A - B$ is positive definite (semidefinite). In particular, $A \succ 0$ ($A \succeq 0$) means that A is a Hermitian positive definite (semidefinite) matrix. If an HPD matrix $X \in \mathbb{C}^{n\times n}$ satisfies $A \preceq X \preceq B$, then we may also write it as $X \in [A, B]$. See [2, 3, 5] for more details about these matrix orderings.

2. The HPD Solutions

In this section, we will investigate existence, uniqueness and some essential properties of the HPD solutions of the NME(1.1).

To simplify discussion, we let

$$Y = Q^{-\frac{1}{2}}XQ^{-\frac{1}{2}}, \quad B = Q^{-\frac{1}{2}}AQ^{-\frac{1}{2}} \quad \text{and} \quad P = Q^{-1}, \tag{2.1}$$

where $Q^{\frac{1}{2}}$ denotes the HPD square root of the HPD matrix Q and $Q^{-\frac{1}{2}} := (Q^{\frac{1}{2}})^{-1}$. Then the NME(1.1) can be equivalently rewritten as the nonlinear matrix equation

$$Y + B^*Y^{-1}PY^{-1}B = I. \tag{2.2}$$

We remark that only when $P = I$, i.e., $Q = I$, the NMEs (1.1) and (2.2) describe the same type of nonlinear matrix equations. Otherwise, they are substantially different from each other. However, as these two equations possess intrinsic relationships and their solutions are internally connected through the matrix transformations (2.1), we can investigate the properties of the solutions of the NME(1.1) by the aid of the NME(2.2).

2.1 The HPD solutions of the NME(2.2)

To have an intuitive understanding of the HPD solutions of the NME(2.2), we first investigate its simplest case when $n = 1$. Now, the NME(2.2) obviously reduces to the form

$$b^2p = y^2 - y^3. \tag{2.3}$$

Define

$$\varphi(y) = y^2 - y^3.$$

As

$$\max_{y\in[0,1]} \varphi(y) = \varphi(\frac{2}{3}) = \frac{4}{27},$$

we know that a necessary condition about the existence of a positive root of the nonlinear equation (2.3) is

$$b^2p \le \frac{4}{27}.$$

Hence, in general, we may assume that the matrices B and P involved in the NME(2.2) satisfy

$$\|B\|^2\|P\| \le \frac{4}{27}.$$