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ON STABLE PERTURBATIONS OF THE STIFFLY WEIGHTED PSEUDOINVERSE AND WEIGHTED LEAST SQUARES PROBLEM *1)

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Abstract

In this paper we study perturbations of the stiffly weighted pseudoinverse $(W^{\frac{1}{2}}A)^{\dagger}W^{\frac{1}{2}}$ and the related stiffly weighted least squares problem, where both the matrices A and W are given with W positive diagonal and severely stiff. We show that the perturbations to the stiffly weighted pseudoinverse and the related stiffly weighted least squares problem are stable, if and only if the perturbed matrices $\widehat{A} = A+\delta A$ satisfy several row rank preserving conditions.

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Key words: Stiffly, Weighted pseudoinverse, Weighted least squares, Perturbation, Stability.

1. Introduction

Consider the following stiffly weighted least squares (stiffly WLS) problem

$$\min_{x\in C^n} \|W^{\frac{1}{2}}(Ax-b)\| = \min_{x\in C^n} \|D(Ax-b)\| \tag{1}$$

and related weighted pseudoinverse [12]

$$A_W^{\dagger} \equiv (W^{\frac{1}{2}}A)^{\dagger}W^{\frac{1}{2}} \text{ with } A_W = WA(WA)^{\dagger}A, \tag{2}$$

where $A \in C^{m\times n}$ with $\text{rank}(A) = r$, $b \in C^m$ are known coefficient matrix and observation vector, respectively, $\|\cdot\| \equiv \|\cdot\|_2$ is the Euclidian vector norm or subordinate matrix norm,

$$D = \text{diag}(d_1, d_2, \cdots, d_m) = \text{diag}(w_1^{\frac{1}{2}}, w_2^{\frac{1}{2}}, \cdots, w_m^{\frac{1}{2}}) = W^{\frac{1}{2}} \tag{3}$$

is the weight matrix. The stiffly WLS problem Eq. (1) with extremely ill-conditioned weight matrix W, where the scalar factors $w_1, \cdots, w_m$ vary widely in size, arises from many areas of applied science, such as in electronic network, certain classes of finite element problems, interior point methods for constrained optimization (e.g., see [8]), and for solving the equality constrained least squares problem by the method of weighting [9, 1, 11], etc.

The stability conditions of the stiffly weighted pseudoinverse and the stiffly WLS problem are important subjects in both theoretical and computational point of view. Wei [11, 12, 13] studied the stability of weighted pseudoinverses and constrained weighted pseudoinverses when the weight matrix W ranges over a set $\mathcal{D}$ of positive diagonal matrices, and obtained necessary and sufficient stability conditions:

$$\textit{if and only if any } r \textit{ rows of the matrix } A \textit{ are linearly independent.} \tag{4}$$

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Based on these results, Wei and De Pieero [16] obtained stability conditions and upper perturbation bounds of WLS and equality constrained least squares problems when weight matrices W range over $\mathcal{D}$.

In practical scientific computations, however, the above condition is too restrictive, and the weight matrix W is usually fixed and severely stiff. In [14], the author found that in this case, the stiffly weighted pseudoinverse is close to a related multi-level constrained pseudoinverse $A_C^\dagger$ and the solution set of Eq. (1) is close to a related multi-level constrained least squares (MCLS) problem. Based on the findings in [14], in this paper we will derive the stability conditions of the stiffly weighted pseudoinverse and the stiffly WLS problem.

Without loss of generality, we make the following notation and assumptions for the matrices A and W.

Assumption 1.1. *The matrices A and W in Eq. (1) satisfy the following conditions: $\|A(i,:)\|$ have the same order for $i = 1, \cdots, m$, $w_1 > w_2 > \cdots > w_k > 0$, $m_1 + m_2 + \cdots + m_k = m$, and we denote*

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_k \end{pmatrix} \begin{matrix} m_1 \\ \vdots \\ m_k \end{matrix}, \quad C_j = \begin{pmatrix} A_1 \\ \vdots \\ A_j \end{pmatrix}, \quad j = 1, \cdots, k, \tag{5}$$

$$\begin{aligned} &W = \mathrm{diag}(w_1 I_{m_1}, w_2 I_{m_2}, \cdots, w_k I_{m_k}), \\ &0 < \epsilon_{ij} \equiv \tfrac{w_i}{w_j} \ll 1, \ \textit{for } 1 \le j < i \le k \ \textit{so } \epsilon = \max_{1 \le j < k} \{\epsilon_{j+1,j}\} \ll 1. \end{aligned} \tag{6}$$

We also set

$$P_0 = I_n, \quad P_j = I - C_j^\dagger C_j, \quad \mathrm{rank}(C_j) = r_j, \quad j = 1, \cdots, k. \tag{7}$$

The paper is organized as follows. In §2 we will review some preliminary results related to the weighted pseudoinverse; in §3 we will study stability conditions for the stiffly weighted pseudoinverse; in §4 we will deduce perturbation bounds for the solutions of the stiffly WLS problem Eq. (1); in §5 we will provide several numerical examples to verify our findings; finally in §6 we will conclude the paper with some remarks.

2. Preliminaries

In this section we provide some preliminary results which are necessary for our further discussion.

Lemma 2.1. [4] *Suppose that $D, E \in C^{m\times n}$ and* $\mathrm{rank}(D) = \mathrm{rank}(E)$. *Then*

$$\begin{aligned} \|DD^\dagger - EE^\dagger\| &\le \min\{\|(D-E)D^\dagger\|, \|(D-E)E^\dagger\|, 1\}, \\ \|D^\dagger D - E^\dagger E\| &\le \min\{\|D^\dagger(D-E)\|, \|E^\dagger(D-E)\|, 1\}. \end{aligned} \tag{8}$$

Lemma 2.2. [14] *Under the notation of Assumption 1.1,*

$$\begin{aligned} &(A_j P_{j-1})^\dagger A_j P_{j-1} = C_j^\dagger C_j - C_{j-1}^\dagger C_{j-1}, \\ &\mathrm{rank}(A_j P_{j-1}) = \mathrm{rank}(C_j) - \mathrm{rank}(C_{j-1}) = r_j - r_{j-1} \end{aligned} \tag{9}$$

for $j = 2, \cdots, k$. Denote $(A_j P_{j-1})^H = Q_j R_j$ the unitary decomposition of $(A_j P_{j-1})^H$ (A_j^H is the conjugate transpose of the matrix A_j), where $Q_j^H Q_j = I_{r_j - r_{j-1}}$ and R_j has full row rank $r_j - r_{j-1}$. Then for $j = 1, \cdots, k$,

$$(Q_1, \cdots, Q_j)^H (Q_1, \cdots, Q_j) = I_{r_j}, \ C_j^\dagger C_j = \sum_{l=1}^{j} Q_l Q_l^H, \tag{10}$$

$$A_j P_{j-1} = A_j Q_j Q_j^H, \ (A_j P_{j-1})^\dagger = Q_j (A_j Q_j)^\dagger. \tag{11}$$