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ARTIFICIAL BOUNDARY METHOD FOR THE THREE-DIMENSIONAL EXTERIOR PROBLEM OF ELASTICITY*1)

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Abstract

The exact boundary condition on a spherical artificial boundary is derived for the three-dimensional exterior problem of linear elasticity in this paper. After this boundary condition is imposed on the artificial boundary, a reduced problem only defined in a bounded domain is obtained. A series of approximate problems with increasing accuracy can be derived if one truncates the series term in the variational formulation, which is equivalent to the reduced problem. An error estimate is presented to show how the error depends on the finite element discretization and the accuracy of the approximate problem. In the end, a numerical example is given to demonstrate the performance of the proposed method.

Mathematics subject classification: 65N30
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1. Introduction

Numerical approximation to the solutions of PDEs in unbounded domains has attracted much attention of engineers and mathematicians in the last three decades. Many effective and efficient methods have been proposed for different problems arising from various research areas. Among them is the so-called artificial boundary method. The key point of this method is to limit the computational domain by introducing a proper artificial boundary in the exterior unbounded domain and imposing a suitable boundary condition on the artificial boundary, to ensure the well-posedness of the reduced problem.

Engquist and Majda [7], Bayliss and Turkel [4] considered first-order hyperbolic equations and other wave-like equations; Han and Wu [19], Yu [25] designed various types of artificial boundary conditions for the exterior Laplace equation; Feng [8], Goldstein [13], Deakin and Rasmussen [6] obtained the nonreflecting boundary conditions for reduced wave equation; Halpern and Schatzman [16], Han and Bao [17] discussed the incompressible flow in a channel; Grote and Keller [14], Alpert, Greengard and Hagstrom [1] considered the exterior problem of time-dependent hyperbolic equation.

For linear elastic problem, Givoli and Keller [11], Han and Wu [19, 20] designed artificial boundary condition on a circular artificial boundary for two-dimensional case. In addition, Han and Bao gave an error analysis in [18] for this problem. For the time-harmonic elastic wave in two dimensions, Givoli and Keller [12] derived the artificial boundary conditions. Grote

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and Keller [15] obtained the exact nonreflecting boundary conditions for elastic waves in three dimensions.

In this paper, we concentrate on the exterior problem of elasticity in three dimensions. We obtain an exact boundary condition on the spherical artificial boundary. This boundary condition is usually called D_tN mapping or D_tN artificial boundary condition. This approach has been used by Han and Wu to solve the exterior Laplace equation and elastic problem in two dimensions (see formulation 2.10 on page 181 of [19]). In the paper by Givoli and Keller [11], they presented the D_tN artificial boundary condition for the exterior problem of Laplace equation in three-dimension and linear elastic problem in two dimensions. For more information on this approach, the reader is referred to the review papers by Givoli [9, 10] and Tsynkov [24].

This paper is organized as follows. In section 2, some results on vectorial spherical harmonics in [21] are listed. In section 3, an exact artificial boundary condition is designed on the spherical artificial boundary. The equivalent variational problem to the reduced problem is formulated in section 4. In section 5, the error analysis is presented. This error estimate is dependent not only on finite element discretization, but on the accuracy of approximate variational formulation. A numerical example is presented in section 6 to show the performance of our method. This paper concludes in section 7.

2. Some Results on Vectorial Spherical Harmonics

It is well-known that the spherical harmonic functions $\{Y_l^m, l \geq 0, -l \leq m \leq l\}$ constitutes an orthogonal basis of space $L^2(S)$, where S denotes the surface of unit sphere (see page 24 in [21]). Let $\mathbf{x}$ be the location vector, $r = |\mathbf{x}|$ and $H_l^m = r^l Y_l^m$, then $\{H_l^m, -l \leq m \leq l\}$ constitutes a basis of all l-order homogeneous harmonic polynomials. We define

$$\begin{aligned}
\mathcal{I}_l^m &\equiv \nabla H_{l+1}^m, \quad l \geq 0, \quad -(l+1) \leq m \leq l+1, \\
\mathcal{T}_l^m &\equiv \nabla H_l^m \times \mathbf{x}, \quad l \geq 1, \quad -l \leq m \leq l, \\
\mathcal{N}_l^m &\equiv (2l-1) H_{l-1}^m \mathbf{x} - r^2 \nabla H_{l-1}^m, \quad l \geq 1, \quad -(l-1) \leq m \leq (l-1)
\end{aligned}$$

and denote by $\mathbf{I}_l^m$, $\mathbf{T}_l^m$ and $\mathbf{N}_l^m$ the traces of these functions on S, i.e.

$$\mathbf{I}_l^m = \frac{\mathcal{I}_l^m}{r^l}, \ \mathbf{T}_l^m = \frac{\mathcal{T}_l^m}{r^l}, \ \mathbf{N}_l^m = \frac{\mathcal{N}_l^m}{r^l}.$$

These functions are called l-order *vectorial spherical harmonics*.

Lemma 2.1. *Let* $\mathbf{n} = \dfrac{\mathbf{x}}{r}$ *be the unit vector in the radial direction, then the following hold*

$$\begin{aligned}
&\nabla\left(\nabla \cdot \frac{\mathbf{I}_l^m}{r^{l+1}}\right) = \frac{(l+1)(2l+1)}{r^{l+3}} \mathbf{N}_{l+2}^m, \\
&\nabla\left(\nabla \cdot \frac{\mathbf{T}_l^m}{r^{l+1}}\right) = 0, \\
&\nabla\left(\nabla \cdot \frac{\mathbf{N}_l^m}{r^{l+1}}\right) = 0, \\
&\left(\nabla \cdot \frac{\mathbf{I}_l^m}{r^{l+1}}\right)\mathbf{n} = -\frac{1}{r^{l+2}}\left\{\frac{(l+1)(2l+1)}{2l+3}\mathbf{I}_l^m + \frac{(l+1)(2l+1)}{2l+3}\mathbf{N}_{l+2}^m\right\}, \\
&\left(\nabla \cdot \frac{\mathbf{T}_l^m}{r^{l+1}}\right)\mathbf{n} = 0, \\
&\left(\nabla \cdot \frac{\mathbf{N}_l^m}{r^{l+1}}\right)\mathbf{n} = 0,
\end{aligned}$$