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THE GENERALIZED MAXIMUM ANGLE CONDITION FOR THE $\mathcal{Q}_1$ ISOPARAMETRIC ELEMENT [*1)]

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Abstract

We consider the quadrilateral $\mathcal{Q}_1$ isoparametric element and establish an optimal error estimate in H^1 norm for the interpolation operator under a weaker mesh condition which admits anisotropic quadrilaterals and allows the quadrilateral to become a regular triangle in the sense of maximum angle condition [5, 11].

Mathematics subject classification: 65N30.
Key words: Quadrilateral mesh, $\mathcal{Q}_1$ isoparametric element, Generalized maximum angle condition

1. Introduction

We shall consider the quadrilateral $\mathcal{Q}_1$ element and establish an estimate for the interpolation error under a new mesh condition. This condition is weaker than the precede conditions proposed in [12] and [2] among others. Moreover, it allows the quadrilateral to degenerate into an anisotropic however regular triangle in the sense of maximum angle condition [5, 11, 2]. First we will review some known results and introduce some notations.

Let K be a convex quadrilateral with vertices M_1, M_2, M_3 and M_4. Let $\hat{K} = [-1, 1]^2$ be the reference element. There exists a bijection mapping $\mathcal{F}_K : \hat{K} \rightarrow K$ that $K = \mathcal{F}_K(\hat{K})$.

Let $\hat{\mathcal{Q}}_1(\hat{K})$ be the bilinear polynomial space, and let $\mathcal{Q}_1 = \mathcal{Q}_1(K)$ be the corresponding space defined on K. Let Π_1 denote the usual bilinear interpolation operator.

Our aim is to obtain the following interpolation error estimate

$$\|u - \Pi_1 u\|_{0,K} + h \mid u - \Pi_1 u \mid_{1,K} \leq C_e h^2 \mid u \mid_{2,K} \tag{1.1}$$

under the condition we shall proposed, where h is the diameter of K. There are several conditions in the literature for (1.1) to hold, here we only review, among others, the J condition and *RDP* condition, proposed by Jamet [12] and Acosta and Duran [2] respectively, which can be expressed as follows

Definition 1.1 *K is regular with constant $\sigma > 0$, or shortly $J(\sigma)$, if it holds that*

$$h/\rho \leq \sigma,$$

where h denotes the diameter of K and ρ the maximum of the diameters of all circles contained in K.

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Definition 1.2 *K is regular with constant $N \in R$ and $0 < \psi < \pi$, or shortly RDP(ψ,N), if we can divide K into two triangles along one of its diagonals, which will always be called D_1, the other is D_2 in such a way that $\mid D_2 \mid / \mid D_1 \mid \leq N$ and both triangles satisfy the maximum angle condition, i.e., each interior angle of these two triangles is bounded from above by ψ.*

For other conditions, we referr to references [7, 8, 9, 14] and [3, 17]. A comprehensive review of quadrilateral meshes can be found in the introduction of [14], there the equivalency and the relation of some shape mesh conditions is also proved. The review of degenerate quadrilateral mesh conditions can also be found in [2].

Under the *J(σ)* condition, it was shown in [12] that the constant C_e in (1.1) depends only on σ. Under the constraint *RDP(ψ, N)*, Acosta and his colleague prove that C_e depends only on ψ and N. *RDP(ψ, N)* condition is so far the weakest mesh condition for (1.1) to hold. However, due to the constraint $\mid D_2 \mid / \mid D_1 \mid \leq N$, it does not allow a quadrilateral to become an anisotropic however regular triangle in the sense of maximum angle condition. As we will see below the constraint $\mid D_2 \mid / \mid D_1 \mid \leq N$ can be removed.

We introduce some notations and concepts. Let d_1 denote the longer diagonal of K, d_2 the shorter one. As illustrated in Fig.1, we denote by T_1 and T_2 the two triangles obtained by subdividing K along d_1, and t_1 and t_2 are the two triangles obtained by decomposing K along d_2.

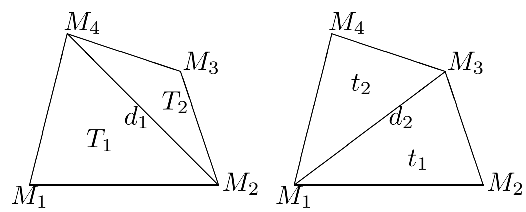

Fig.1. Quadrilateral K

We now give the definition of the maximum angle condition.

Definition 1.3 *[5, 11, 2] We say a triangle T (resp. a quadrilateral K) satisfies the maximum angle condition with a constant ψ, or shortly MAC(ψ), if the angles of T (resp. K) are less than or equal to ψ.*

In the sequel, the regularity of triangles is refered to as in this maximum angle sense. Our mesh condition can be stated as

Definition 1.4 *We say a convex quadrilateral K satisfies the generalized maximum angle condition, or shortly GMAC(ψ), if there exists a positive constant $\psi < \pi$ such that, among T_i, t_i, $i = 1, 2$, there are at least three regular triangles in the sense of MAC(ψ).*

Let us notice that the constraint $\mid D_2 \mid / \mid D_1 \mid \leq N$ in the *RDP(N, ψ)* condition is dropped in this condition. We shall prove the following result

Theorem 1.1 *Let K be a convex quadrilateral satisfying GMAC(ψ) with the constant $0 < \psi < \pi$ and $u \in H^2(K)$, then there exists a constant C_{err} only depending on ψ such that*

$$\mid u - \Pi_1 u \mid_{m,K} \leq C_{err}(\psi) h^{2-m} \mid u \mid_{2,K}, m = 0, 1 \tag{1.2}$$