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P_1-NONCONFORMING QUADRILATERAL FINITE VOLUME ELEMENT METHOD AND ITS CASCADIC MULTIGRID ALGORITHM FOR ELLIPTIC PROBLEMS *

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Abstract

In this paper, we discuss the finite volume element method of P_1-nonconforming quadrilateral element for elliptic problems and obtain optimal error estimates for general quadrilateral partition. An optimal cascadic multigrid algorithm is proposed to solve the non-symmetric large-scale system resulting from such discretization. Numerical experiments are reported to support our theoretical results.

Mathematics subject classification: 65N30, 65N55.
Key words: finite volume element method, cascadic multigrid, Elliptic problems.

1. Introduction

Finite volume method(FVM) is a discretization technique widely used in the approximation of conservation laws, in computational fluid dynamics, and in convection-diffusion problems. Apart from an approximation of the solution at discrete points, we can seek a discrete solution in a finite element space. This version of approximation is often called the finite volume element method(FVEM). On the one hand, it has a simplicity for implementation comparable to the finite difference method and can be viewed as a generalization of the finite difference method; on the other hand, it has a flexibility similar to that of the finite element method(FEM) for handling complicated geometries and boundary conditions and preserves more mathematical structures of the original continuous problem, which makes systematic error analysis possible. Another important advantage of this method is that such generated numerical solutions usually have certain conservation property locally, thus it can be expected to capture shocks, to produce simple stencils, or to study other physical phenomena more effectively. About its recent developments, we refer to the monographs [2, 9, 10, 11, 14, 18, 22] for details.

Nonconforming elements have been used effectively especially in the computation of fluid and solid mechanics due to their stability nature. Recently increasing attentions have been paid to these elements for their potential application in parallel computing. Driven by these reasons, many nonconforming elements have been proposed in the triangular and quadrilateral cases from 1970s [13, 15, 16, 20, 21]. Observing the fact that any P_1 function on a quadrilateral can be uniquely determined by its values on any three of the four midpoints on the edges, [20] and [16] introduced the P_1-nonconforming quadrilateral element from different points of view and this element has the least degrees of freedom among all the low order nonconforming quadrilateral elements. The quadrilateral finite element spaces are generally constructed starting from a given

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finite dimensional polynomials space $\hat{V}$ on a reference element $\hat{K}$ by a bilinear isomorphism. Recent observation made by [1] implies that for such defined finite element spaces, a necessary and sufficient condition for approximation of order $r+1$ in L^p and r in $W^{1,p}$ is that $\hat{v}$ contains the space Q_r. Thus for the truly quadrilateral element, the P_1-nonconforming finite element space obtained from the standard reference element will not guarantee the optimal convergence rate anymore. But the nonparametric scheme proposed in [20] provides an efficient way of computing without losing the order of convergence.

In this paper, we are interested in using the nonparametric P_1-nonconforming quadrilateral element to solve elliptic problems by FVEM. Considering the particular characteristic of this element, we propose its finite volume element discretization scheme corresponding to a dual partition of overlapping type. Numerical analysis shows optimal convergence rate under H^1-norm, but in order to obtain optimal error estimate under L^2-norm, additional assumptions on the source term and the partition are needed. A counterexample is given to show that more regular assumption on the source term is necessary. But numerical experiments demonstrate that the assumption on the partition is unnecessary, which means the L^2-norm error estimate may can be improved.

In the field of scientific computing, designing effective algorithm to solve the systems resulting from the discretization of PDEs is always the concern of many researchers. Cascadic multigrid method, which requires no coarse grid corrections and can be viewed as a "one-way" multigrid method, is proven to be effective for solving large-scale finite element discretization problem, see [3, 4, 5, 6, 7, 17, 24, 25, 27] for details. But for the finite volume element discretization, the algebraic systems of self-adjoint elliptic problems are nonsymmetric in general, which brings many difficulties for designing optimal cascadic multigrid algorithms. Based on the observation that the nonsymmetric equations are a small perturbation of the usual finite element discretization equations, we propose a new cascadic multigrid algorithm in [26] to solve the finite volume element discretization problem for P_1-conforming triangular element. The aim of this paper is to apply this algorithm to the P_1-nonconforming quadrilateral element. The nonconformity of this element is conquered by defining a new inter-grid transfer operator. Theoretical analysis and numerical experiments show that this algorithm is optimal in both accuracy and computational complexity.

The rest of our paper is organized as follows: In Section 2, we give some notations used in this paper and formulate the FVE scheme for the nonparametric P_1-nonconforming quadrilateral element for self-adjoint elliptic problems; then in Section 3, we obtain optimal H^1- and L^2-norm error estimates for it, and a counterexample is given to show that the L^2-norm error estimate cannot be optimal in regularity; Section 4 is devoted to analyze the cascadic multigrid algorithm for the discretization problem; then in the last section, we give some numerical experiments to support our theoretical results.

2. Notations and the Finite Volume Element Scheme for the P_1-nonconforming Quadrilateral Element

In this paper, we consider the following self-adjoint elliptic problem

$$\begin{aligned} -\nabla\cdot(\mathbf{A}\nabla u) &= f, \quad in\ \Omega, \\ u &= 0, \quad on\ \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a convex polygonal domain in R^2, and $\mathbf{A}=(a_{i,j})_{2\times 2}\in(W^{1,\infty}(\Omega))^4$ is a given real matrix function satisfying

$$0<\alpha_*|\xi|^2\le\xi^t\mathbf{A}(x)\xi\le\alpha^*|\xi|^2<\infty, \qquad \forall\xi\in R^2. \tag{2.2}$$

In what follows we shall adopt the standard definitions of Sobolev spaces, the notations of their norms and semi-norms as presented in [12].