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# SPECTRAL APPROXIMATION ORDERS OF MULTIDIMENSIONAL NONSTATIONARY BIORTHOGONAL SEMI-MULTIRESOLUTION ANALYSIS IN SOBOLEV SPACE *1)

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**Abstract**

Subdivision algorithm (Stationary or Non-stationary) is one of the most active and exciting research topics in wavelet analysis and applied mathematical theory. In multidimensional non-stationary situation, its limit functions are both compactly supported and infinitely differentiable. Also, these limit functions can serve as the scaling functions to generate the multidimensional non-stationary orthogonal or biorthogonal semi-multiresolution analysis (Semi-MRAs). The spectral approximation property of multidimensional non-stationary biorthogonal Semi-MRAs is considered in this paper. Based on nonstationary subdivision scheme and its limit scaling functions, it is shown that the multidimensional nonstationary biorthogonal Semi-MRAs have spectral approximation order $r$ in Sobolev space $H^s(\mathbb{R}^d)$, for all $r \geq s \geq 0$.

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## 1. Introduction

Subdivision algorithm, resulting from several fields of applied mathematics and signal processing, is an iterative method to generate smooth curves and surfaces. For example, to construct planar curves, such a scheme begins with the initial control points $f_0(k)$ defined on the integer lattice $\mathbb{Z}$, and then expends the control points to the fine lattice $\mathbb{Z}/2 := \{j/2 | j \in \mathbb{Z}\}$ via a specified mask $h_{j,k} = \{h_{j,k}(l)\}_{l\in\mathbb{Z}}$. Usually, we assume that the mask $h_{j,k}$ is a finite sequence, i.e. for every $j \geq 0$ and each $k \in \mathbb{Z}$, the set $\{l| \in \mathbb{Z}, h_{j,k}(l) \neq 0\}$ only contains finite elements. After $j$ iterative steps, it derives a new sequence $f_j(2^{-j}k)$. The iterative procedure satisfies the following linear rule:

$$f_j(2^{-j}k) = 2 \sum_{n\in k+2\mathbb{Z}} h_{j,k}(n) f_{j-1}(2^{-j}(k-n))\,. \qquad (1.1)$$

If mask $h_{j,k}$ is independent of both scale $j$ and position $k$, namely $h_{j,k}(l) = h_l$, then this subdivision scheme is said to be stationary, otherwise to be nonstationary. In the case of stationary subdivision algorithm, (1.1) can be rewritten as:

$$f_j(2^{-j}k) = 2 \sum_{n} h_{k-2n} f_{j-1}(2^{-j+1}n)\,.$$

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The convergence of above stationary subdivision scheme is closely connected with the existence of the solution to the refinement equation as follows.

$$f(x) = 2\sum_{n\in\mathbb{Z}} h_n f(2x-n)\,.$$

Thereby, the stationary subdivision schemes play an important role in the wavelet theory [7, 10, 11, 12, 13].

However, stationary multiresolution analysis based on a compactly supported refinable function is limited to generators (scaling functions) with a finite degree of smoothness. So, one cannot build a $C^\infty$ refinable function which is also compactly supported in stationary case.

More recently, attention has been given to nonstationary subdivision schemes [3, 4, 5, 6]. Since the masks may vary from different scale $j$ or different position $k$, it is possible to construct a nonstationary Semi-MRA which is generated by $C^\infty$ compactly supported scaling functions. In fact, by virtue of Rvachev[8] up-function method, N. Dyn and A. Ron[4] constructed a compactly supported scaling function in $C^\infty$ and the corresponding nonstationary Semi-MRA $\{V_j\}_{j\geq 0}$. The constructed scaling function $\varphi_j(x)$ is defined in the Fourier domain by

$$\hat{\varphi}_j(\omega) = \prod_{k=1}^{+\infty}\left(\frac{1+e^{-i2^{-k}\omega}}{2}\right)^{k+j}, \quad j\geq 0, \tag{1.2}$$

The length of its support is $L_j = \sum_{k\geq 1}(k+j)2^{-k} = j+2$. The scaling space is defined as:

$$V_j := \mathrm{Span}\{\varphi_j(2^j x - k)\}_{k\in\mathbb{Z}}$$

From equation (1.2), it yields that

$$\hat{\varphi}_j(\omega) = m_{j+1}(\omega/2)\hat{\varphi}_{j+1}(\omega/2)\,, \tag{1.3}$$

where

$$m_{j+1}(\omega/2) = \left(\frac{1+e^{-i\omega}}{2}\right)^{j+1}.$$

It also concludes from (1.3) that the spaces $V_j$ are embeded, namely,

$$V_j \subset V_{j+1}, \quad \text{for all } j\geq 0.$$

In addition, the investigation of the spectral approximation order in $L^2$ or Sobolev space is also gaining considerable attention because of its powerful theoretical analysis for approximation theory. Encouraging results have been reported in some literatures [4, 5], [14]-[17]. More details, the paper [4] showed that its constructed nonstationary Semi-MRA $\{V_j\}_{j\geq 0}$ has spectral approximation property in $L^2(\mathbb{R})$, i.e., for all $r\geq 0$ and $f(x)\in H^r(\mathbb{R})$, $\lim_{j\to+\infty} 2^{jr}\|P_j f - f\|_0 = 0$. Cohen and Dyn [5] exploited a technique introduced in [14] to generalize these results to some nonstationary subdivision schemes in one dimensional case. de Boor, DeVore and Ron [14] are concerned with approximation in the $L^2$ norm from shift-invariant spaces. Cohen and Dyn [5] adapted their technique to the derivation of density orders in Sobolev norms. Approximation orders in Sobolev norms by shift-invariant spaces are studied in paper [15] and [16]. Yoon [17] considered the spectral approximation orders in Sobolev space using radial basis function interpolation.

In paper [18], we previously obtained some results on the convergence of multidimensional nonstationary subdivision algorithm and properties of its limit functions. We also exploited these results to generate multidimensional nonstationary biorthogonal Semi-MRAs [19]. The goal of this paper is to prove that the multidimensional nonstationary biorthogonal Semi-MRAs constructed in [19] have spectral approximation order $r$ in Sobolev space $H^s(\mathbb{R}^d)$.

To this end, some multi-index notations are given as follows:

- Multi-index $m=(m_1,\cdots,m_d)\in\mathbb{N}_0^d$, $|m| := m_1+\cdots+m_d$;