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HIGH RESOLUTION SCHEMES FOR CONSERVATION LAWS AND CONVECTION-DIFFUSION EQUATIONS WITH VARYING TIME AND SPACE GRIDS *1)

Hua-zhong Tang
(*LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China*)

Gerald Warnecke
(*Institut für Analysis und Numerik, Otto-von-Guericke Universität Magdeburg, 39106 Magdeburg, Germany*)

Abstract

This paper presents a class of high resolution local time step schemes for nonlinear hyperbolic conservation laws and the closely related convection–diffusion equations, by projecting the solution increments of the underlying partial differential equations (PDE) at each local time step. The main advantages are that they are of good consistency, and it is convenient to implement them. The schemes are L^∞ stable, satisfy a cell entropy inequality, and may be extended to the initial boundary value problem of general unsteady PDEs with higher–order spatial derivatives. The high resolution schemes are given by combining the reconstruction technique with a second order TVD Runge-Kutta scheme or a Lax-Wendroff type method, respectively.

The schemes are used to solve a linear convection–diffusion equation, the nonlinear inviscid Burgers' equation, the one– and two–dimensional compressible Euler equations, and the two–dimensional incompressible Navier–Stokes equations. The numerical results show that the schemes are of higher–order accuracy, and efficient in saving computational cost, especially, for the case of combining the present schemes with the adaptive mesh method [15]. The correct locations of the slow moving or stronger discontinuities are also obtained, although the schemes are slightly nonconservative.

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Key words: Hyperbolic conservation laws, Degenerate diffusion, High resolution scheme, Finite volume method, Local time discretization.

1. Introduction

This paper is aimed at the construction of numerical approximations for nonlinear hyperbolic conservation laws

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = 0, \tag{1.1}$$

and the closely related convection–diffusion equations

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = \frac{\partial}{\partial x}Q(u, u_x), \tag{1.2}$$

with given initial data $u(x,0) = u_0(x)$ and corresponding suitable boundary conditions. Here $u = u(x,t)$ is an m–vector of conserved quantities, $x \in \mathbb{R}^d$, $m, d \geq 1$, $f(u) = \big(f^1(u), \cdots, f^d(u)\big)$

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is a nonlinear convective flux vector, and $Q(u, u_x) = \big(Q^1(u, u_x), \cdots, Q^d(u, u_x)\big)$ is a dissipation flux vector satisfying the weak parabolicity condition $\nabla_v Q(u, v) \geq 0$, for all u and v in $\mathbb{R}^m$.

These equations are of great practical importance since they arise in fluid flows, for example, reactive flows, groundwater flows, non–Newtonian flows, traffic flows, two-phase flows in oil reservoirs etc. During the past few decades there has been an enormous amount of activity related to the construction of high resolution schemes for Eqs.(1.1) and (1.2), see for instance [4, 12] and references therein. Explicit high resolution methods have been proven to be very efficient in capturing moving discontinuities or fronts, such as shock waves etc. However, they need a small time step size satisfying a Courant-Friedrichs-Lewy (CFL) condition for Eq.(1.1) and a similar more restriction condition for Eq.(1.2), to guarantee stability. For an implicit scheme, the time step size is also often constrained by nonlinear convergence. The spatial step sizes and the "signal" speeds are the two main elements to limit a choice of the time step size. Hence, when solving numerically unsteady PDEs, it may occur that in some spatial regions there is the need for a smaller time step than in other regions. Typical examples are numerical simulations of viscous fluid flows on nonuniform meshes and other computations of solutions to PDEs on an adaptive mesh [3, 9, 15]. Due to the above reason, the large time step schemes [7, 18] and the local time step schemes [3, 5, 9] become attractive. The large time step schemes satisfy the CFL condition by automatically increasing the stencil with the size of the time step. They can give correctly the location of shocks with virtually no smearing, but they seem to be inconvenient in practical applications, especially in treating boundary conditions.

The local time step schemes are only restricted by a local stability condition rather than the traditional global stability condition dominated by the smallest cells. The schemes studied in [3, 5, 9] are conservative, but they suffer a loss of consistency near a time grid interface in terms of truncation errors, see Section 2. Recently, the local time step schemes are widely studied and extended to adaptive grid methods, see e.g. [8, 14, 16].

The discrete conservation of a numerical algorithm for (1.1) or (1.2) is important in order to keep the correct location of the discontinuities. Hou and LeFloch in [6] have shown that if a nonconservative scheme for (1.1) converges, it converges to a solution of $\partial_t u + \partial_x f(u) = \mu$, where μ is a Borel measure source term that is expected to be zero in the region where the solution u is smooth and concentrated where u is not smooth. Even so, nonconservative schemes are also valuable in some practical applications and have been implemented successfully, for example, in computations of compressible multi-fluids [2] and fluid flows on an overlapping grid [10]. Another kind of the nonconservative schemes are residual distribution methods for hyperbolic conservation laws [1].

The aim of this paper is to study high resolution local time discretization schemes for (1.1) and (1.2), by projecting the solution increments of the underlying PDEs at each local time step. Because of good consistency, they may be applied to solving the initial boundary value problem of general unsteady PDEs with higher–order spatial derivatives. Moreover, the schemes are L^∞ stable, and it is convenient to implement them. Although the schemes will loose locally the discrete conservation, correct shocks have been obtained numerically when computing the 1D scalar Burgers' equation and the cylindrical explosion problem of the Euler equations.

This paper is organized as follows. In Section 2, we first review and analyze the scheme of Osher and Sanders [9]. A simple projection of the solutions is used in their scheme, but it suffers a loss of local consistency with the governing equations. Motivated by their scheme, we present a class of high resolution local time step schemes for Eq. (1.1) by projecting directly the solution increments of the underlying PDEs at each local time step. They include consistent first order Euler type schemes, second–order Runge–Kutta type schemes, and second–order Lax–Wendroff type schemes with multi–time increments. The schemes suffer from a slight loss of conservativity at isolated time grid interfaces. It turns out though, that shock speeds are hardly perturbed, see our numerical computations in Section 4. Because of good consistency, we may extend them to solve unsteady PDEs with higher–order derivatives, for example, Eq. (1.2). In Section 3, our