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THE EIGENVALUE PERTURBATION BOUND FOR ARBITRARY MATRICES *1)

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Abstract

In this paper we present some new absolute and relative perturbation bounds for the eigenvalue for arbitrary matrices, which improves some recent results. The eigenvalue inclusion region is also discussed.

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1. Introduction

Let A be an $n \times n$ matrix and $\widetilde{A} = A + E$ whose spectrum are $\{\lambda_1, ..., \lambda_n\}$ and $\{\mu_1, ..., \mu_n\}$, respectively. Let $\|\cdot\|_F$ and $\|\cdot\|_2$ denote the Frobenius norm and the spectral norm, respectively. For a positive integer n, let $\langle n \rangle = \{1, 2, ..., n\}$.

Classical absolute type perturbation bounds were established by the well-known Hoffman-Wielandt theorem [1]. When A and $\widetilde{A}$ are normal matrices, there exists a permutation τ of $\langle n \rangle$ such that

$$\sqrt{\sum_{i=1}^{n} |\mu_{\tau(i)} - \lambda_i|^2} \leq \|E\|_F. \tag{1.1}$$

In the case that A is normal but $\widetilde{A}$ is arbitrary, Sun proved [2,3] that there exists a permutation τ of $\langle n \rangle$ such that

$$\sqrt{\sum_{i=1}^{n} |\mu_{\tau(i)} - \lambda_i|^2} \leq \sqrt{n}\|E\|_F. \tag{1.2}$$

The factor $\sqrt{n}$ in (1.2) is optimal in some sense [2]. Furthermore, Song [4] studied the more general case. For two arbitrary matrices, he obtained

$$\sqrt{\sum_{i=1}^{n} |\mu_{\tau(i)} - \lambda_i|^2} \leq \sqrt{n}(1 + \sqrt{n-p}) \max\left\{\|Q^{-1}EQ\|_F,\ \|Q^{-1}EQ\|_F^{1/m}\right\} \tag{1.3}$$

and

$$|\mu_{\tau(i)} - \lambda_i| \leq \sqrt{n}(1 + \sqrt{n-p}) \max\left\{\|\sqrt{n}Q^{-1}EQ\|_2,\ \|\sqrt{n}Q^{-1}EQ\|_F^{1/m}\right\} \tag{1.4}$$

where $Q^{-1}AQ = diag(J_1, ..., J_p)$ defines the Jordan form of A and m is the order of the largest Jordan block.

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As well known for any matrix $\widetilde{A} \in \mathbf{C}^{n\times n}$, there is a unitary matrix U such that $U^*\widetilde{A}U = diag(\widetilde{A}_1, ..., \widetilde{A}_s)$, where $\widetilde{A}_i$ is an upper triangular matrix, $i = 1, ..., s$. In [5] the authors showed that if A is normal then for any matrix $\widetilde{A}$ there exists a permutation τ such that

$$\sqrt{\sum_{i=1}^{n} |\mu_{\tau(i)} - \lambda_i|^2} \leq \sqrt{n-s+1}||E||_F. \tag{1.5}$$

It is noted that s in (1.5) is not unique. In fact, it need not to decompose $\widetilde{A}$ so that $\widetilde{A}_i$ is upper triangular. Now let $\widetilde{A} \in \mathbf{C}^{n\times n}$, we denote by $s(\widetilde{A})$

$$s(\widetilde{A}) = \max_{U \text{ is unitary}} \{q : U^*\widetilde{A}U = diag(\widetilde{A}_1, ..., \widetilde{A}_q),\ \widetilde{A}_i \text{ is square},\ i = 1, ..., q\}. \tag{1.6}$$

This means that $s(\widetilde{A})$ is the most diagonal block numbers for which $\widetilde{A}$ is unitarily similar to a block diagonal matrix. Hence $s(\widetilde{A})$ exists and is unique for any matrix, and for any unitary matrix Q, $s(Q^*\widetilde{A}Q) = s(\widetilde{A}) \geq 1$. Notice if $\widetilde{A}$ is normal, then $s(\widetilde{A}) = n$.

By (1.5) it is easy to prove the following result.

Theorem 1.1. *Let $s(\widetilde{A})$ be given by (1.6), and let A be normal. Then for any matrix $\widetilde{A}$ there exists a permutation τ such that*

$$\sqrt{\sum_{i=1}^{n} |\mu_{\tau(i)} - \lambda_i|^2} \leq \sqrt{n-s(\widetilde{A})+1}||E||_F. \tag{1.7}$$

In this paper, we shall improve the bounds in (1.3) and (1.4) for arbitrary matrices $\widetilde{A}$ and A based on Theorem 1.1. The relative bound and the eigenvalue inclusion region are also considered.

2. The Absolute Bound

First we write A into its Jordan canonical form

$$Q^{-1}AQ = J = diag(J_1, ..., J_p), \tag{2.1}$$

where J_i be an $m_i \times m_i$ Jordan block matrix with the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix},\ i = 1, ..., p.$$

For $\varepsilon \neq 0$, let

$$T = diag(T_1, ..., T_p),\ T_i = diag(1, \varepsilon, \varepsilon^2, ..., \varepsilon^{m_i-1}), i = 1, ..., p. \tag{2.2}$$

Then from (2.1) it is easy to check that

$$T^{-1}Q^{-1}AQT = \Lambda' + \Delta'_\varepsilon,$$

where $\Lambda' = diag(\lambda_1 I_{m_1}, ..., \lambda_p I_{m_p})$, $\Delta'_\varepsilon = diag(\Delta'_1, ..., \Delta'_p)$ and

$$\Delta'_i = \begin{bmatrix} 0 & \varepsilon & & 0 \\ & 0 & \ddots & \\ & & \ddots & \varepsilon \\ 0 & & & 0 \end{bmatrix}_{m_i\times m_i}, \quad i = 1, ..., p.$$