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SPURIOUS NUMERICAL SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS *1)

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Abstract

This paper deals with the relationship between asymptotic behavior of the numerical solution and that of the true solution itself for fixed step-sizes. The numerical solution is viewed as a dynamical system in which the step-size acts as a parameter. We present a unified approach to look for bifurcations from the steady solutions into spurious solutions as step-size varies.

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1. Introduction

It is well-known that a numerical method which is convergent in a finite interval does not necessarily yield the same asymptotic behavior as the underlying differential equation. In many circumstances, we are interested in the asymptotic behavior in the differential equations. The asymptotic states of a dynamical system are captured in the $\omega-$ and $\alpha-$ limit sets which may concern equilibria, periodic orbits, attractors, etc. It is desirable to design numerical schemes for which these sets are close to the corresponding limit sets of the underlying differential equation, and to understand and hence to avoid conditions under which spurious members of the limit sets are introduced by the time discretization.

Runge-Kutta and linear multistep methods are commonly used to obtain a numerical solution of ordinary differential equations (ODEs). Dynamics of the numerical solution produced by Runge-Kutta and linear multistep methods solving ODEs has been extensively studied (see, for example, [3, 6, 7, 8, 9, 10, 12, 17]).

In this paper, we are concerned with the nonlinear delay differential equation with a constant lag in the form

$$\begin{aligned} y'(t) &= f(y(t), y(t-\tau)), \quad t > 0, \\ y(t) &= \phi(t), \quad -\tau \le t \le 0, \end{aligned} \tag{1}$$

where y, f are real scalar functions and $\tau > 0$ is a constant lag. The solution (if it exists) is determined by a choice of initial function ϕ. The results on existence, uniqueness and continuous dependence of solution of (1) can be found in the books by Hale and Lunel [4] and Driver [2]. We assume throughout that the initial function ϕ is continuous.

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Consider approximating the solution of (1) using a consistent numerical method with a fixed step-size h such that $h = \frac{\tau}{m} > 0$, where $m > \widetilde{k}$ is some positive integer and $\widetilde{k}$ is a positive integer depending upon the specific method. Let y_n denote our approximation to $y(t_n)$, where $t_n = nh$. Typically the sequence y_n is derived from a map of the form

$$\mathcal{F}(y_{n-m}, \cdots, y_{n+\widetilde{k}}; h) = 0, \quad n = 0, 1, \ldots, \tag{2}$$

together with $\widetilde{k}$ initial conditions. Thus (2) must be solved for $y_{n+\widetilde{k}}$ given $y_{n-m}, \cdots, y_{n+\widetilde{k}-1}$. By introducing a new vector $U_n = [y_{n-m}, \cdots, y_{n+\widetilde{k}-1}]^T \in \mathbb{R}^{\widetilde{k}+m}$ we may write (2) as a one-step map of the form

$$\mathcal{H}(U_n, U_{n+1}; h) = 0. \tag{3}$$

Definition 1.1.

1. *The numerical scheme (2) is regular of degree* 1*, denoted* $R^{[1]}$*, if every fixed point* $\widehat{u} \in \mathbb{R}$ *of (2) satisfies* $f(\widehat{u}, \widehat{u}) = 0$ *of (1) for all* $h > 0$ *and all equations (1) with* $f \in C^2$*. Otherwise it is irregular of degree* 1*.*

2. *The numerical scheme (2) is regular of degree* 2*, denoted* $R^{[2]}$*, if (2) does not admit real period two solution in* n *for all* $h > 0$ *and all equations (1) with* $f \in C^2$*. Otherwise it is irregular of degree* 2*.*

The following two lemmas are used in the proofs of our main results. The first one concerns the bifurcation of fixed points from simple eigenvalue, while the second concerns the bifurcation of period 2 solutions in the map (3).

Lemma 1.2 [7] *Let the function* $\mathcal{H}(a, b; h)$ *satisfy* $C^r(\mathbb{R}^{\widetilde{k}+m} \times \mathbb{R}^{\widetilde{k}+m}, \mathbb{R})$ *for some integer* $r \geq 2$*. Assume that the map (3) has a fixed point* $\widehat{U}$ *for all* $h > 0$*. Assume also that* $\frac{\partial \mathcal{H}}{\partial a}(\widehat{U}, \widehat{U}; h) + \frac{\partial \mathcal{H}}{\partial b}(\widehat{U}, \widehat{U}; h)$ *is singular at* $h = h_c$ *and there exists a nonzero vector* $\eta \in \mathbb{R}^{\widetilde{k}+m}$ *such that* $Null(\frac{\partial \mathcal{H}}{\partial a}(\widehat{U}, \widehat{U}; h_c) + \frac{\partial \mathcal{H}}{\partial b}(\widehat{U}, \widehat{U}; h_c)) = span\{\eta\}$*. If*

$$\frac{d}{dh}\left(\frac{\partial \mathcal{H}}{\partial a}(U, U; h) + \frac{\partial \mathcal{H}}{\partial b}(U, U; h)\right)\Big|_{U=\widehat{U},\ h=h_c} \eta \notin Range\left(\frac{\partial \mathcal{H}}{\partial a}(\widehat{U}, \widehat{U}; h_c) + \frac{\partial \mathcal{H}}{\partial b}(\widehat{U}, \widehat{U}; h_c)\right).$$

Then, for $0 < \epsilon \ll 1$*, there exists a fixed point of (3) with the form*

$$\begin{aligned} h(\epsilon) &= h_c + \mathcal{O}(|\epsilon|), \\ U_n(\epsilon) &= \epsilon\eta + \mathcal{O}(|\epsilon|^2) \end{aligned}$$

which is C^{r-1} *in* ϵ*.*

Lemma 1.3 [7] *Let the function* $\mathcal{H}(a, b; h)$ *satisfy* $C^r(\mathbb{R}^{\widetilde{k}+m} \times \mathbb{R}^{\widetilde{k}+m}, \mathbb{R})$ *for some integer* $r \geq 2$*. Assume that the map (3) has a fixed point* $\widehat{U}$ *for all* $h > 0$*. Assume also that there exists a nonzero vector* $\vartheta \in \mathbb{R}^{\widetilde{k}+m}$ *such that* $Null\left(\frac{\partial \mathcal{H}}{\partial a}(\widehat{U}, \widehat{U}; h_c) - \frac{\partial \mathcal{H}}{\partial b}(\widehat{U}, \widehat{U}; h_c)\right) = span\{\vartheta\}$ *and that* $\frac{\partial \mathcal{H}}{\partial a}(\widehat{U}, \widehat{U}; h_c) + \frac{\partial \mathcal{H}}{\partial b}(\widehat{U}, \widehat{U}; h_c)$ *is invertible. If*

$$\frac{d}{dh}\left(\frac{\partial \mathcal{H}}{\partial a}(U, U; h) - \frac{\partial \mathcal{H}}{\partial b}(U, U; h)\right)\Big|_{U=\widehat{U},\ h=h_c} \vartheta \notin Range\left(\frac{\partial \mathcal{H}}{\partial a}(\widehat{U}, \widehat{U}; h_c) - \frac{\partial \mathcal{H}}{\partial b}(\widehat{U}, \widehat{U}; h_c)\right).$$

Then, for $0 < \epsilon \ll 1$*, there exists a period* 2 *solution of (3) with the form*

$$\begin{aligned} h(\epsilon) &= h_c + \mathcal{O}(|\epsilon|), \\ U_n(\epsilon) &= \widehat{U} + \epsilon(-1)^n\vartheta + \mathcal{O}(|\epsilon|^2) \end{aligned}$$

which is C^{r-1} *in* ϵ*.*