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A PARALLEL NONOVERLAPPING DOMAIN DECOMPOSITION METHOD FOR STOKES PROBLEMS *

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Abstract

A nonoverlapping domain decomposition iterative procedure is developed and analyzed for generalized Stokes problems and their finite element approximate problems in $\mathbf{R}^N$(N=2,3). The method is based on a mixed-type consistency condition with two parameters as a transmission condition together with a derivative-free transmission data updating technique on the artificial interfaces. The method can be applied to a general multi-subdomain decomposition and implemented on parallel machines with local simple communications naturally.

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Key words: Generalized Stokes problem, Nonoverlapping, Domain decomposition, Mixed finite element, Crouzeix-Raviart element.

1. Introduction

It is known that large scale simulation of viscous incompressible fluid flow requires the solution of the nonlinear time-dependent Navier-Stokes equations. The key and most time-consuming part of this process is the solution of the generalized Stokes problem at each nonlinear iteration. The numerical solution of the generalized Stokes problem plays a fundamental role in the simulation of viscous incompressible fluid flow. Therefore, efficient algorithms for the generalized Stokes problem are indispensable for the numerical solution of the Navier-Stokes equations. However, the systems arising from the generalized Stokes problem are indefinite. This then causes difficulty for solving systems by most preconditioner and iterative methods. On the other hand, it is also difficult to solve these systems directly since they are very large usually. Therefore, a nonoverlapping domain decomposition iterative procedure seems to be attractive for such ill-conditioned problems because it combines iterative methods on the artificial interfaces method on the small subdomains.

The motivation of this work is to develop and analyze a nonoverlapping domain decomposition iterative procedure for solving the generalized Stokes problem and its finite element approximate problems. The nonoverlapping domain decomposition iterative procedure is based on a mixed-type consistency condition with two parameters as a transmission condition together with a derivative-free transmission date updating technique on the artificial interfaces. The method can be applied in a general multi-subdomain decomposition and implemented on parallel machines with local multi-subdomain decomposition and implemented on parallel machines with local simple communications naturally. Firstly ,we consider the nonoverlapping domain decomposition method for the differential problems of generalized Stokes problem. In particular, its convergence is demonstrated by a "pseudo energy" technique. Then, we apply the method to the famous Crouzeix-Raviart linear nonconforming finite element problems of

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generalized Stokes problem. A nonoverlapping domain decomposition iterative procedure is developed for solving the Crouzeix-Raviart linear nonconforming finite element problems. Its convergence is proved for a very general domain decomposition and finite element mesh even without the quasi-uniform and regular requirements. The algorithm is directly presented to the finite element problem without introducing any Lagrange multipliers.

Nonoverlapping domain decomposition methods have been studied extensively and become very attractive for their parallelism and flexibility (cf. [4-7,9-11,15-21]) . The basic idea to develop our method is originally from [5] as well as [6]. A mixed-type transmission condition with two parameters and its derivative-free updating technique are developed and analyzed for second order elliptic problems. In [6], the idea of [5] is extended and applied into mixed finite element problems for second order elliptic problems and mixed finite element methods of nearly elastic waves in frequency domain as well. The other closely related works are [4,11,16-21], In particular, the word of [4] is very similar to [15] but with only one parameter in the transmission condition. In fact, the method in [4] uses the same transmission data as the famous Lions method of [15] but different updating techniques. Moreover, the method of [4] can be regarded as a variant and improvement of Lions method for the continuous differential problems. In [4,11,16-21], the Lions method of [15] is applied into the generalized Stokes problem and its closely related problems, for instance, the Oseen equations, as well as their finite element approximations. All of the rest apply the Lions method to the various problems, for example, the mixed finite element problem (cf.[10]), by introducing Lagrange multipliers on the artificial interfaces.

2. Generalized Stokes Problem and Its Finite Element Approximations

Let Ω be a domain of $\mathbf{R}^N (N{=}2,3)$ and $\partial\Omega$ its boundary. For the sake of simplicity, this paper is to consider the following generalized Stokes problem over Ω.

$$\begin{cases} -\Delta u + \alpha u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $u = (u_1(x), \cdots, u_N(x))$ is the velocity vector, $p = p(x)$ is pressure function, the $f \in \mathbf{L}^2(\Omega) \equiv [L^2(\Omega)]^N$ is the field of external forces, and α is a non-negative constant either 0 or $\alpha_0 > 0$. When $\alpha \equiv 0$ we have the Stokes problem, and the case $\alpha \equiv \alpha_0 > 0$ usually arises as part of the solution process for the Navier-Stokes equations or non-stationary Stokes equations by implicit difference discrete for time (cf.[12],[19]).

The most commonly used Galerkin-type weak formula for the generalize Stokes problem (2.1) is: Find $(u,p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{cases} a(u,v)_\Omega + b(v,p)_\Omega = (f,v)_\Omega \quad \forall v \in \mathbf{H}_0^1(\Omega), \\ b(u,q)_\Omega = 0 \quad \forall q \in L_0^2(\Omega), \end{cases} \tag{2.2}$$

where $\mathbf{H}_0^1(\Omega) = [H_0^1(\Omega)]^N$, $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ whose integrable function with zero mean value, $(\cdot\ ,\ \cdot)_\Omega$ inner product over $L^2(\Omega)$ or $[L^2(\Omega)]^N$

$$a(u,v)_\Omega = (\nabla u, \nabla v)_\Omega + (\alpha u, v)_\Omega, \tag{2.3}$$

$$b(v,q)_\Omega = (q, \nabla \cdot v)_\Omega. \tag{2.4}$$

It is well-known that the generalized Stokes problem (2.1)-(2.2) has a unique solution $(u,p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$. Moreover, there holds the regularity $(u,p) \in (\mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)) \times (L_0^2(\Omega) \cap H^1(\Omega))$