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IMPLEMENTATION OF MIXED METHODS AS FINITE DIFFERENCE METHODS AND APPLICATIONS TO NONISOTHERMAL MULTIPHASE FLOW IN POROUS MEDIA *

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

In this paper we consider mixed finite element methods for second order elliptic problems. In the case of the lowest order Brezzi-Douglas-Marini elements (if $d = 2$) or Brezzi-Douglas-Durán-Fortin elements (if $d = 3$) on rectangular parallelepipeds, we show that the mixed method system, by incorporating certain quadrature rules, can be written as a simple, cell-centered finite difference method. This leads to the solution of a sparse, positive semidefinite linear system for the scalar unknown. For a diagonal tensor coefficient, the sparsity pattern for the scalar unknown is a five point stencil if $d = 2$, and seven if $d = 3$. For a general tensor coefficient, it is a nine point stencil, and nineteen, respectively. Applications of the mixed method implementation as finite differences to nonisothermal multiphase, multicomponent flow in porous media are presented.

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1. Introduction

We consider mixed finite element approximations of the model elliptic problem

$$\begin{aligned} -\nabla \cdot (K\nabla p) &= f \quad \text{in } \Omega, \\ K\nabla p \cdot \nu &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^d$, $d = 2$ or 3, K is a symmetric, positive definite tensor with components in $L^\infty(\Omega)$, ν is the outer unit normal to the domain, and f satisfies the compatibility condition $(f, 1) = 0$ (let $(\cdot\,,\,\cdot)_S$ denote the $L^2(S)$ inner product; we omit S if $S = \Omega$). In applications to flow in porous media, p is the pressure, $u = -K\nabla p$ is the velocity field, and K is the permeability tensor.

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Problem (1.1) is recast in mixed form as follows: Let

$$H(\mathrm{div};\Omega) = \left\{v \in \left(L^2(\Omega)\right)^d : \nabla \cdot v \in L^2(\Omega)\right\},$$
$$V = \{v \in H(\mathrm{div};\Omega) : v \cdot \nu = 0 \text{ on } \partial\Omega\},$$
$$W = L^2(\Omega).$$

Then the mixed form of (1.1) for the pair $(u,p) \in V \times W$ is

$$\begin{aligned} (\nabla \cdot u, q) &= (f,q), && \forall q \in W, \\ (K^{-1}u, v) - (p, \nabla \cdot v) &= 0, && \forall v \in V. \end{aligned} \tag{1.2}$$

To define a finite element method, we need a partition $\mathcal{E}_h$ of Ω into elements E, say, simplexes, rectangular parallelepipeds, and/or prisms, where adjacent elements completely share their common edge or face. Let $V_h \times W_h \subset V \times W$ denote some standard mixed finite element space for second order elliptic problems defined over $\mathcal{E}_h$ (see, e.g., [5, 6, 7, 13, 16, 18]). Then the mixed finite element solution of (1.2) is $(u_h, p_h) \in V_h \times W_h$ satisfying

$$\begin{aligned} (\nabla \cdot u_h, q) &= (f,q), && \forall q \in W_h, \\ (K^{-1}u_h, v) - (p_h, \nabla \cdot v) &= 0, && \forall v \in V_h. \end{aligned} \tag{1.3}$$

The mixed method (1.3) requires the solution of a linear system in the form of a saddle point problem, which can be expensive to solve. An alternate approach was suggested by means of a nonmixed formulation. Namely, it is shown that the mixed finite element method is equivalent to a modification of the nonconforming Galerkin method [1, 3, 8, 12, 15]. The nonconforming method yields a symmetric and positive definite problem (i.e., a minimization problem). In the case that K is a diagonal tensor and one uses the lowest order Raviart-Thomas-Nédélec [16, 18] spaces over a rectangular grid, it is shown [19] that the linear system arising from the mixed formulation can be simplified by use of certain quadrature rules. That is, the mixed method system can be written as a cell-centered finite difference method.

An analogous simplification of the mixed method system as a finite difference method for another widely used space, the lowest order Brezzi-Douglas-Marini space [7] if $d = 2$ or the lowest order Brezzi-Douglas-Durán-Fortin [5] space if $d = 3$ has not been known. The objective of this paper is to derive a finite difference method for this space, without any loss in the rate of convergence, and retaining the superconvergence result. In particular, we show that for a diagonal tensor coefficient, the lowest order Brezzi-Douglas-Marini mixed method can be written as a cell-centered finite difference method with a five point stencil, and the Brezzi-Douglas-Durán-Fortin method can be given with a nine point stencil. For a general tensor coefficient, these two methods can be written with a nine point stencil, and nineteen, respectively. Our approach illuminates a relationship between the lowest order Raviart-Thomas-Nédélec and Brezzi-Douglas-Marini (or Brezzi-Douglas-Durán-Fortin) spaces; i.e., they can be written as the same finite difference method for the pressure by an appropriate use of quadrature rules.

The rest of the paper is organized as follows. In §2, we rewrite (1.2) using numerical quadrature rules for the evaluation of the integrals on each element $E \in \mathcal{E}_h$, and prove solvability. Then, in §3 we derive the finite difference method for the Brezzi-Douglas-Marini and Brezzi-Douglas-Durán-Fortin spaces. In §4, we mention some convergence and superconvergence results. In §5, we address another difficulty that the permeability K can be in practice zero in a subset of Ω. We consider an expanded mixed formulation in the sense that three variables are explicitly treated, i.e., the pressure, the velocity field, and the flux field. This new formulation can handle the difficulty arising from the zero permeability [2, 9, 10, 11]. Finally, applications of the mixed method implementation as finite differences to nonisothermal multiphase, multicomponent flow in porous media are presented.