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OPTIMAL ERROR ESTIMATES OF THE PARTITION OF UNITY METHOD WITH LOCAL POLYNOMIAL APPROXIMATION SPACES *1)

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

In this paper, we provide a theoretical analysis of the partition of unity finite element method(PUFEM), which belongs to the family of meshfree methods. The usual error analysis only shows the order of error estimate to the same as the local approximations[12]. Using standard linear finite element base functions as partition of unity and polynomials as local approximation space, in 1-d case, we derive optimal order error estimates for PUFEM interpolants. Our analysis show that the error estimate is of one order higher than the local approximations. The interpolation error estimates yield optimal error estimates for PUFEM solutions of elliptic boundary value problems.

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Key words: Meshless methods, Partition of unity finite element method(PUFEM), Error estimate.

1. Introduction

As a new family of numerical methods, in the last few years meshless methods came into the focus of interest, especially in the engineering community. This is motivated by the often encountered serious difficulties in generating meshes for problems in complex domains, or in domains evolving with the problem solution. In addition, a need to have flexibility in the selection of approximating functions (e.g., the flexibility to use non-polynomial approximating functions), played a significant role in the development of meshless methods. A recent survey of meshless and generalized finite element methods was given by [2] together with a comprehensive list of references. It states the development in this new field and provides the available mathematical theory with proofs. From its list of references, we can learn that more and more interest has been directed towards an important subclass of methods originating from the partition of Unity Method(PUM) of *Babuška* and Melenk [3]. These methods include the hp cloud method of Oden and Duarte [5], the Generalized Finite Element Method (GFEM) of Strouboulis [14]-[16] and the particle-partition of unity method (see [6]-[9] and [11]). Applying the partition of unity Huang and Xu[11] proposed a conforming finite element method for overlapping nonmatching

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grids. Why is the PUFEM so popular? The most prominent reasons are that PUFEM can include a priori knowledge about the local behavior of the solution in the finite element space and construct finite element spaces of any desired regularity. In general, we only know that the function u of interest is in some function space. As [1] points out, the local approximating spaces of PUFEM have many function spaces including polynomials and non-polynomial functions to choose for the solutions to a given differential equation. These function spaces are not unique and the choice of particular space thus depends on practical aspects (cost of constructing the functions, ease of evaluation of the functions, i.e. cost of construction of the stiffness matrix; conditioning number of the resulting stiffness matrix) and theoretical aspects (optimality of the function space). It is well known the $h-$, $p-$, and $hp-$ versions of the finite element method (FEM) use local polynomial approximating as shape functions. The success of FEM is due to the fact that a smooth function can be approximated locally by polynomials and polynomial spaces are big enough to absorb extra constraints of continuity cross interelement boundaries without loosing the approximation properties. Therefore, for sufficiently smooth function, polynomial space is preferable for the local approximating spaces in PUFEM. If the usual piecewise linear hat functions are taken as a (M, C_∞, C_G) partition of unity and the local approximation spaces are chosen to be spaces of polynomials, the PUFEM then can be referred to a generalization of the $h-$ and $p-version$ FEM. And the PUFEM have approximation properties very similar to the usual $h-$ and $p-version$ FEM [4]. Some general results on the PUFEM are provided in [3] by using technique of Taylor expansion. *Babuška* and his co-workers give main theoretical results about GFEM in [4]. Recently a posteriori estimation for GFEM similar to that for FEM can be found in [13].

The usual error analysis only shows the order of error estimate to the same as the local approximations[12]. Using standard linear finite element base functions as partition of unity and polynomials as local approximation space, in 1-d case, we derive optimal order error estimates for PUFEM interpolants. Our analysis will show that the error estimate is of one order higher than the local approximations, that is, global error estimate of order $p+1$ is achieved while the local error estimates on patches is only of order p. For this purpose, we construct a special polynomial local approximation space according to the consistence and local approximation properties of PUFEM at first, and then we derive the interpolation error estimation of PUFEM by employing the arguments in [10] and applying various techniques of Taylor expansion and theories of average polynomials interpolation. The interpolation error estimates are used to obtain optimal order error estimates for PUFEM solutions of Neumann boundary value problems. We will also show how to derive optimal order error estimates for PUFEM solutions of Dirichlet boundary value problems (BVPs) in one dimension. The error estimates we establish in this paper are in the one dimensional setting and under sufficient smoothness assumption on the functions being approximated. Error analysis for singular problems and the higher dimensional case will be addressed in the forthcoming papers.

The paper is organized as follows. In section 2, we provide a precise introduction of PUFEM, emphasizing mathematical foundation behind the development of the method. A kind of special local approximation space based on polynomials will be constructed and optimal order error estimates for PUFEM interpolants are then established in section 3. In section 4, we discuss error estimates for PUFEM solutions of boundary value problems.

2. Mathematical Foundation of the PUFEM

In this section, we present a method of constructing conforming subspaces of $H^1(\Omega)$. We construct finite element spaces which are subspaces of $H^1(\Omega)$ as an example because of their importance in applications. We would like to point out that the method leads to the construction of smoother spaces (subspaces of $H^k(\Omega), k > 1$) or subspaces of Sobolev spaces $W^{k,p}$ in a straightforward manner. we introduce the main concepts and results concerning the error