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TWO-SCALE FINITE ELEMENT DISCRETIZATIONS FOR PARTIAL DIFFERENTIAL EQUATIONS *[1)]

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

Some two-scale finite element discretizations are introduced for a class of linear partial differential equations. Both boundary value and eigenvalue problems are studied. Based on the two-scale error resolution techniques, several two-scale finite element algorithms are proposed and analyzed. It is shown that this type of two-scale algorithms not only significantly reduces the number of degrees of freedom but also produces very accurate approximations.

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Key words: Finite element, Two-scale discretization, Parallel computation, Sparse grids.

1. Introduction

It is a challenging task to solve 3−dimensional ($3d$) partial differential equations by conventional discretization methods, due to storage requirements and computational complexity. Usually, both storage requirements and running time grow tremendously when the number of degrees of freedom for approximate solutions increases. Thus, for $3d$ applications such as problems from computational materials science, computational chemistry and computational biology, the most elaborate solver routines like multigrid or multilevel methods should be applied in order to obtain numerical solutions with satisfactory accuracy. Additionally, the code should be implemented on a high-performance computer.

To reduce the computational cost, including the computational time and the storage requirement, some new two-scale finite element discretizations for solving partial differential equations in $3d$ are introduced in this paper. The main idea of our new discretizations is to use a coarse grid to approximate the low frequencies and to combine some univariate fine and coarse grids to handle the high frequencies by some parallel procedures. These discretizations are based on our understanding of the frequency resolution of a finite element solution to some elliptic problem. For a solution to an elliptic problem, it is shown that low frequency components can be approximated well on a relatively coarse grid and high frequency components can be computed on a fine grid (see, e.g., [4, 17, 25, 31]). It is also observed that for elliptic problems on tensor product domains, a part of high frequencies results from the tensor product of the univariate low frequencies, which can then be damped out by the tensor product of some fine and coarse grids.

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We now give a somewhat more detailed but informal (and hopefully informative) description of the main ideas and results in this paper. Consider an elliptic boundary value problem in domain $\Omega = (0,1)^3$. Let $P_{h_{x_1},h_{x_2},h_{x_3}}u$ be the standard trilinear finite element solution, that is, the Ritz-Galerkin approximation, of a partial differential equation on a uniform grid $T^{h_{x_1},h_{x_2},h_{x_3}}$ with mesh size h_{x_1} in x_1-direction, h_{x_2} in x_2-direction and h_{x_3} in x_3-direction, respectively. Then, a two-scale finite element approximation, which is nothing but a simple combination of different standard finite element solutions of the original problem over different scale meshes, is constructed as follows (see Section 3):

$$P^h_{H,H,H}u \equiv P_{h,H,H}u + P_{H,h,H}u + P_{H,H,h}u - 2P_{H,H,H}u,$$

where $H \gg h$.

In this two-scale approximate scheme, only partially refined meshes are involved, and the following result for a class of partial differential equations can be established (see Theorem 3.1)

$$\|u - P^h_{H,H,H}u\|_{1,\Omega} = O(h + H^2), \tag{1.1}$$

where u is the exact solution of the partial differential equation.

This is a very satisfactory result in many ways. Consequently, for example, we obtain an asymptotically optimal approximation $P^h_{H,H,H}u$ in parallel by taking $H = O(\sqrt{h})$ and the number of degrees of freedom for obtaining $P^h_{H,H,H}u$ is only of $O(h^{-2})$, while that for the standard finite element solution $P_{h,h,h}u$ with the same approximate accuracy is of $O(h^{-3})$.

We may also design efficient two-scale approximate schemes for other problems. For instance, consider the following eigenvalue problem posed on Ω:

$$\begin{cases} -\nabla(a\nabla u) &= \lambda u, \ \text{in} \quad \Omega, \\ \qquad u &= 0, \ \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where a is a positive smooth function on $\bar{\Omega}$. We may employ the following algorithm to approximate (1.2) (see Section 4):

1. Solve (1.2) on a coarse grid: find $(u_{H,H,H}, \lambda_{H,H,H}) \in S_0^{H,H,H}(\Omega) \times R^1$ such that $\int_\Omega a|\nabla u_{H,H,H}|^2 = 1$ and
$$\int_\Omega a\nabla u_{H,H,H}\nabla v = \lambda_{H,H,H}\int_\Omega u_{H,H,H}v, \quad \forall v \in S_0^{H,H,H}(\Omega). \tag{1.3}$$
2. Compute the linear boundary value problems on partially fine grids in parallel: find $u^{h,H,H} \in S_0^{h,H,H}(\Omega)$ such that
$$\int_\Omega a\nabla u^{h,H,H}\nabla v = \lambda_{H,H,H}\int_\Omega u_{H,H,H}v, \quad \forall v \in S_0^{h,H,H}(\Omega);$$
find $u^{H,h,H} \in S_0^{H,h,H}(\Omega)$ such that
$$\int_\Omega a\nabla u^{H,h,H}\nabla v = \lambda_{H,H,H}\int_\Omega u_{H,H,H}v, \quad \forall v \in S_0^{H,h,H}(\Omega);$$
find $u^{H,H,h} \in S_0^{H,H,h}(\Omega)$ such that
$$\int_\Omega a\nabla u^{H,H,h}v = \lambda_{H,H,H}\int_\Omega u_{H,H,H}v, \quad \forall v \in S_0^{H,H,h}(\Omega).$$
3. Set
$$u^h_{H,H,H} = u^{h,H,H} + u^{H,h,H} + u^{H,H,h} - 2u_{H,H,H}$$
and
$$\lambda^h_{H,H,H} = \frac{\int_\Omega a|\nabla u^h_{H,H,H}|^2}{\int_\Omega |u^h_{H,H,H}|^2},$$