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NEW PROOF OF DIMENSION FORMULA OF SPLINE SPACES OVER T-MESHES VIA SMOOTHING COFACTORS *1)

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Abstract

A T-mesh is basically a rectangular grid that allows T-junctions. Recently, Deng *etal* introduced splines over T-meshes, which are generalizations of T-splines invented by Sederberg *etal*, and proposed a dimension formula based on the B-net method. In this paper, we derive an equivalent dimension formula in a different form with the smoothing cofactor method.

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Key words: Spline space, T-mesh, Smoothing cofactors.

1. Introduction

T-meshes are formed by a set of horizontal line segments and a set of vertical line segments, where T-junctions are allowed. See Figure 1 for examples.

Traditional tensor-product B-spline functions, which are a basic tool in the design of free-form surfaces, are defined over special T-meshes, where no T-junctions appear. B-spline surfaces have the drawback that arises from the mathematical properties of the tensor-product B-spline basis functions. Two global knot vectors which are shared by all basis functions, do not allow local modification of the domain partition. Thus, if we want to construct a surface which is flat in the most part of the domain, but sharp in a small region, we have to use more control points not only in the sharp region, but also in the regions propagating from the sharp region along horizontal and vertical directions to maintain the tensor-product mesh structure. The superfluous control points are a big burden to modelling systems. In [5], Sederberg *etal* explained the troubles made by these superfluous control points in details.

To overcome this limitation, we need the local refinement of B-spline surfaces, i.e. to insert a single control point without propagating an entire row or column of control points. In [4] hierarchical B-splines were introduced, and two concepts were defined: local refinement using an efficient representation and multi-resolution editing. In principle, Hierarchical B-splines are the accumulation of tensor-product surfaces with different resolutions and domains. Weller and Hagen [8] discussed tensor-product splines with knot segments. In fact, they defined a spline space over a more general T-mesh, where crossing, T-junctional, and L-junctional vertices are allowed. But its dimensions are estimated and its basis functions are given over the mesh induced by some semi-regular basis functions.

In 2003, Sederberg *etal* [5] invented T-spline. It is a point-based spline, i.e., for every vertex, a blending function of the spline space is defined. Each of the blending functions comes from some tensor-product spline space. Though this type of splines supports many valuable operations within a consistent framework, but some of them, say, local refinement, are

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not simple. In the T-spline theory, the local refinement is dependent on the structure of the mesh, and its complexity is uncertain. Whether T-spline blending functions are always linearly independent is an open question [6]. The reason leading to these problems is that the spline over every cell of the mesh is not a polynomial, but a piecewise polynomial.

In [2], Deng *etal* formulated the concept of T-meshes, and studied the spline space over T-meshes. They forced the spline on every cell to be a tensor-product polynomial and achieve the specified smoothness across common edges, and derived a dimension formula when the smoothness is less than half of the degree of polynomials with a method based on B-nets.

In the theory of multivariate spline, smoothing cofactor method [7] is another dominant approach to calculate the dimension of some specified spline space. In this paper, we derive a dimension formula equivalent to Deng's formula with the smoothing cofactor method. The proof is longer than the B-net version, but it is revelatory. Based on some results in this paper, we have implemented a quasi-real-time algorithm, which will be explored in another forthcoming paper, to calculate the dimension of a general spline space over T-meshes. And we expect that we can generalize Deng's formula based on the smoothing cofactor method in the future.

The paper is organized as follows. Section 2 presents a brief review of the spline spaces over T-meshes. In Section 3, by introducing the concepts of vertex cofactor and in-line, we derive a dimension formula for the spline space $\mathcal{S}(m, n, \alpha, \beta, \mathcal{T})$ when $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$ with the smoothing cofactor method, and prove that it is equivalent to Deng's formula. In the final section, we conclude the paper with some further research problems.

2. Spline Spaces over T-meshes

In this section, we first present some concepts related with T-meshes, and then review spline function spaces over T-meshes.

2.1 T-mesh

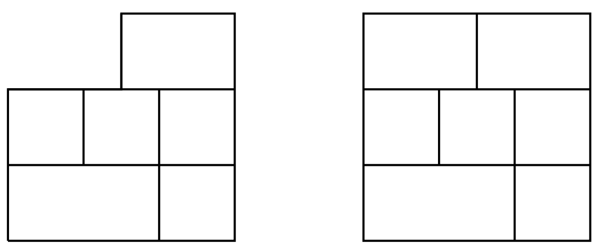

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Figure 1: Examples of T-mesh

A **T-mesh** is basically a rectangular grid that allows T-junctions [5]. The longest possible horizontal or vertical line segments to make up a T-mesh are called **grid lines**. We assume that the endpoints of each grid line in the T-mesh must be on two other grid lines, and each **cell** or **facet** (the area without any line segment inside it) in the grid must be a rectangle. Figure 1 illustrates two examples of T-meshes, while in Figure 2 two examples of non-T-meshes are shown.

A grid point in a T-mesh is also called a **vertex** of the T-mesh. If a vertex is on the boundary grid line of a T-mesh, then is called a **boundary vertex**. Otherwise, it is called an **interior vertex**. For example, b_i, $i = 1, \ldots, 10$ in Figure 3 are boundary vertices, and all the