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A PROJECTION-TYPE METHOD FOR SOLVING VARIOUS WEBER PROBLEMS *1)

Jian-lin Jiang Bo Chen
(*Department of Mathematics, Nanjing University, Nanjing 210093, China*)

Abstract

This paper investigates various Weber problems including unconstrained Weber problems and constrained Weber problems under l_1, l_2 and l_∞-norms. First with a transformation technique various Weber problems are turned into a class of monotone linear variational inequalities. By exploiting the favorable structure of these variational inequalities, we present a new projection-type method for them. Compared with some other projection-type methods which can solve monotone linear variational inequality, this new projection-type method is simple in numerical implementations and more efficient for solving this class of problems; Compared with some popular methods for solving unconstrained Weber problem and constrained Weber problem, a singularity would not happen in this new method and it is more reliable by using this new method to solve various Weber problems.

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Key words: Linear variational inequality, Various Weber problems, Projection-type method, Slack technique.

1. Introduction

Weber problem (WP) is one of the fundamental models in location theory and has many applications in practice, see, e.g., [10]. Its objective is to site a new facility in the plane to minimize a sum of weighted distances from the new facility to a set of customers whose locations are known. Weber problem has the following formulation:

$$\text{WP:} \qquad \min_{x\in R^2} C(x) = \sum_{j=1}^{n} w_j \|x - a_j\|_p, \tag{1.1}$$

where a_i is the known location of the ith customer, $i = 1, \cdots, n$; n is the number of customers; x is the unknown location of the new facility; w_i is the weight associated with the customer a_i, $i = 1, \cdots, n$; $\|\cdot\|_p$ is the distance measuring function.

When the new facility x is restricted to be sited in a constrained area X, this model is named as constrained Weber problem (CWP).

Some efficient methods have been proposed for solving Weber problem and constrained Weber problem. Weiszfeld procedure [12] is perhaps the most popular and standard method for Weber problem with Euclidean distances; Recently, a so-called Newton-Bracketing (NB) method [8] was presented to solve Weber problem. The well-known method for constrained Weber problem whose constrained area is the union of a finite set of convex polygons was presented in [3], which consists in a search for the unconstrained solution followed by an exploration of some of the boundary parts of the polygons defining the feasible region.

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However, a singularity may happen for these popular methods: if an iterate generated by them is identical with one of customers, the next iterate is undefined. The reason for this singularity is that a "bad" initial point is chosen for these methods. Chandrasekaran and Tamir [2] showed that the set of these "bad" initial points may contain a continuum set, and thus in advance we have no way to clearly know whether one initial point is bad or not.

In this paper, we discuss various Weber problems (VWP) including Weber problems and constrained Weber problems under l_1, l_2 and l_∞-norms,

$$\text{VWP:} \qquad \min_{x\in X} C(x) = \sum_{j=1}^{n} w_j \|x - a_j\|_p, \tag{1.2}$$

where X is a constrained area which is closed and convex in R^2. Note that VWP reduce to Weber problem in the case that $X = R^2$. With a transformation technique, various Weber problems can be reformulated as min-max problems from which a class of monotone linear variational inequalities (LVIs) may be obtained,

$$\begin{cases} (x - x^*)^T (A^T z^* + q_1) \ \geq \ 0 & \forall\, x \in X, \\ (z - z^*)^T (-Ax^* + q_2) \ \geq \ 0 & \forall\, z \in Z, \end{cases} \tag{1.3}$$

where $x \in R^k,\ z \in R^{kn},\ A = (I_k, \cdots, I_k)^T \in R^{kn\times k}$, I_k is the $k \times k$ identity matrix and X and Z are closed convex sets. Thus, solving various Weber problems is equivalent to solving (1.3).

Many computational methods have been established for solving monotone linear variational inequality. The projection-type methods, e.g., projection-contraction (PC) methods, may be one class of the simplest methods for solving these problems and they are also applicable for solving (1.3). Our purpose is to exploit the favorable structure of (1.3) in practice and propose a more efficient projection-type method for it. Note that for LVI (1.3) $A^T A = nI_k$. Based on this observation, a new projection-type method is proposed. The new method is rather simple in numerical implementations. The most significance for proposing this new method is that for an arbitrarily chosen initial point the singularity would not happen for this new method, which guarantees that using this method we can acquire the optimal solution of various Weber problems. Numerical results are reported, which shows that the new projection-type method is meaningful for solving these problems.

The paper is organized as follows. Some popular methods for solving Weber problem are provided in Section 2. In Section 3 various Weber problems under l_1, l_2 and l_∞-norms are transformed into this class of variational inequalities (1.3). Some preliminaries required in coming analysis are given in Section 4. The new projection-type method for solving this class of variational inequalities is presented in Section 5. In Section 6 the convergence of the new method is provided and preliminary numerical results are reported in Section 7. Finally, some concluding remarks are drawn in the last section.

2. Some Existing Algorithms for Solving Weber Problem

In this section we discuss two popular methods for solving Weber problem: Weiszfeld procedure and Newton-Bracketing method.

2.1 Weiszfeld procedure

Since the distance measuring function is convex, as a sum of convex functions, the objective function $C(x)$ of Weber problem is convex. It is clear that the set of its optimal solutions is nonempty and convex. Whereas, the main difficulty for solving Weber problem is that $C(x)$ is non-differentiable at some locations, e.g., the locations of customers. The gradient of $C(x)$