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ON THE DIVIDED DIFFERENCE FORM OF FAÀ DI BRUNO'S FORMULA *1)

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Abstract

The n-divided difference of the composite function $h := f \circ g$ of functions f, g at a group of nodes $t_0, t_1, \cdots, t_n$ is shown by the combinations of divided differences of f at the group of nodes $g(t_0), g(t_1), \cdots, g(t_m)$ and divided differences of g at several partial group of nodes $t_0, t_1, \cdots, t_n$, where $m = 1, 2, \cdots, n$. Especially, when the given group of nodes are equal to each other completely, it will lead to Faà di Bruno's formula of higher derivatives of function h.

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1. Introduction

It is studied that a divided difference of a function at a group of given nodes can be shown by combining the divided difference of the same function at another group of nodes[1,6,11]. When the given nodes are closed together, the divided difference is difficult to be computed. If the divided difference can be described by the combination of the divided difference of the same function at another distant node, then can be easily computed [12]. That is to say, it is the generalization of Lagrange numerical derivative method.

It is well-known that the Leibniz formula

$$h^{(n)}(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} f^{(\nu)}(x) g^{(n-\nu)}(x) \tag{1.1}$$

of higher derivative of $h(x) := f(x) \cdot g(x)$, whose divided difference form is as follows:
Steffensen[10] formula. Let $h(x) := f(x) \cdot g(x)$. For any nodes $x_0, x_1, \cdots, x_n$, we have

$$h[x_0, x_1, \cdots, x_n] = \sum_{\nu=0}^{n} f[x_0, x_1, \cdots, x_\nu] \cdot g[x_\nu, x_{\nu+1}, \cdots, x_n]. \tag{1.2}$$

Now, let us begin to study the formulas of divided difference or the form of Faà di Bruno's formula for the composite function. Recently, introductions to the Faà di Bruno's formula of higher derivative of composite function[9, 5] and it's generalizations [3, 7] have been closely noticed by researchers. Johnson [9] stated not only its history, but also its partition description under the view of combination, the Bell polynomial description, determinant description, and various kinds of formulas based on Taylor formulas. One of the aims of this paper is to give

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a supplement to the Johnson's interesting results based on what we obtain in the paper. Of cause, Basic formulas obtained in the paper about the divided difference is undoubtedly useful for numerical analysis and combination analysis.

We will begin from the first order case. It is known that in primary calculus the formulas of the first order derivative for the composite function is as the following:

Chain Rule. Suppose $x = g(t)$ and $y = f(x)$ have derivatives at $t = t_0$ and $x = g(t_0)$, respectively. Then, the composite function $y = h(t) := f(g(t))$ has derivative at $t = t_0$,and

$$h'(t) = f'(x_0)g'(t_0) = f'(g(t_0))g'(t_0). \tag{1.3}$$

The proof of the Chain Rule should be done as followings. Let $t_1 \neq t_0$, and $x_1 = g(t_1)$, we have

$$\frac{h(t_1) - h(t_0)}{t_1 - t_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \cdot \frac{g(t_1) - g(t_0)}{t_1 - t_0}. \tag{1.4}$$

Let $t_1 \to t_0$, (1.3) follows from (1.4) immediately. But the bug of this proof method is that we can not guarantee $x_1 \neq x_0$, even though we can let $t_1 \neq t_0$. So there is something wrong with (1.4). To avoid this tragedy, in Courant & John's well-known calculus course [4], strict condition of $g'(t)$ has no zero point on an interval is used.

As a matter of fact, the case $x = x_1$ is not a tragedy. it just takes $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ into the existed derivative $f'(x_0)$. If we use the following definition of the first order divided difference of f at nodes x_0, x_1:

$$f[x_0, x_1] := \begin{cases} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}, & \text{if } x_1 \neq x_0; \\ f'(x_0), & \text{if } x_1 = x_0, \end{cases} \tag{1.5}$$

then (1.4) can be replaced by the following suitable equality

$$\frac{h(t_1) - h(t_0)}{t_1 - t_0} = f[x_0, x_1]\frac{g(t_1) - g(t_0)}{t_1 - t_0}. \tag{1.6}$$

The above equation surely holds for $x_1 \neq x_0$. As long as $y = f(x)$ has derivative at $x = x_0$. Let $t_1 \to t_0$ in (1.6), then the Chain Rule (1.3) follows.

Besides, we can easily find that (1.3) and the key step in its proof can be unified into the following formula:

$$\begin{aligned} h[t_0, t_1] &= f[x_0, x_1]g[t_0, t_1] \\ &= f[g(t_0), g(t_1)]g[t_0, t_1], \end{aligned} \tag{1.7}$$

which are called as *divided difference form of Chain Rule* or *chain rule of the first order divided difference*. When $t_1 \neq t_0$, (1.7) becomes (1.6); while $t_1 = t_0$, (1.7) becomes (1.3).

Studies in the paper shows that similar results hold for higher divided differences. That is, Faà di Bruno's formula for higher derivative of composite function has its relative form of divided difference. It will be given in the following Theorem 2.2 and 2.3, which not only give a suitable generalization to Faà di Bruno's formula, but also give the simplest proof method for the formula. They are basic formulas about divided difference.

2. Main Results

To discuss the n-th order divided difference of f at a group of nodes $x_0, x_1, \cdots, x_n$, we first give its definition of recursion form for nodes which are different from each other. Then, we understand the divided difference at nodes where some nodes are equal as the limit of the divided difference mentioned above. From this point of view, we have (for example, see Isaacson & Keller [8])