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ON QUADRATURE OF HIGHLY OSCILLATORY FUNCTIONS *1)

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Abstract

Some quadrature methods for integration of $\int_a^b f(x)e^{i\omega g(x)}dx$ for rapidly oscillatory functions are presented. These methods, based on the lower order remainders of Taylor expansion and followed the thoughts of Stetter [9], Iserles and Nørsett [5], are suitable for all ω and the decay of the error can be increased arbitrarily in case that $g'(x) \neq 0$ for $x \in [a, b]$, and easy to be implemented and extended to the improper integration and the general case $I[f] = \int_a^b f(x)e^{ig(\omega,x)}dx$.

Mathematics subject classification: 65D32, 65D30.
Key words: Oscillatory integral, quadrature, Filon-type method, Taylor expansion.

1. Introduction

The quadrature of highly oscillating integrals is important in many areas of applied mathematics. The standard integration formulas such as the trapezoid rule, Simpson's rule or Gaussian integration may suffer from difficulty. Many methods have been developed since Filon [2], such as Price [8], Stetter [9], Longman [6], Levin [7], Iserles [3,4] and Iserles and Nørsett [5], etc.

For the Filon-type quadrature of the form $\int_0^h f(x)e^{i\omega x}w(x)dx$, Iserles [3] analyzed the convergent behavior in a range of frequency regimes and showed that the accuracy increases when oscillation becomes faster. Recently Iserles and Nørsett [5] extended the approach of Iserles [3,4] and defined the generalized Filon-type method for integral $\int_0^1 f(x)e^{i\omega g(x)}dx$ and showed that the rate of decay of the error, once frequency grows, can be increased arbitrarily by the inclusion of higher derivatives.

Both the Filon-type and the generalized Filon-type, an approach f(x) by splines, are efficient for suitably smooth functions under the condition that the moments $\int_0^1 x^k e^{i\omega g(x)}dx$ can be accurately calculated.

Price's numerical approximation of Fourier transforms [8] is considered the integration between the zeros, for example,

$$\int_0^{2\pi} f(x)\sin nx dx.$$

Each

$$\int_{k\pi/n}^{(k+1)\pi/n} f(x)\sin nx dx$$

may be expeditiously computed by use of a Labatto rule. The Price method completely fails when ω is significantly larger than the number of quadrature points.

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Here we present some methods, basing on the lower order remainders of Taylor expansion, which transfer highly oscillatory functions into 'nice' functions—non-highly oscillating functions. And following the thoughts of Setter [9], Iserles and Nørsett [5] we show that these methods are suitable for all ω and the decay of the error can be increased arbitrarily the same as the generalized Filon-type method for large ω in case that $g'(x) \neq 0$ for $x \in [a,b]$. These methods are easy to be implemented and extended to the improper integration and the integral

$$I[f] = \int_a^b f(x) e^{ig(\omega,x)} dx,$$

where $e^{ig(\omega,x)}$ with highly oscillation and $\lim_{\omega\to\infty} g'_x(\omega,x) = \infty$ for all x in $[a,b]$.

2. Quadrature of Integral $\int_a^b f(x)e^{i\omega g(x)}dx$

Let $I[f]$ denote the following integral

$$I[f] = \int_a^b f(x) e^{i\omega g(x)} dx, \tag{2.1}$$

where f and g are suitably smooth functions. Suppose that the function g has at most finite stationary points in $[a,b]$. Without loss of generality, assume g has only one stationary point x_0 in $[a,b]$. Otherwise, we will partition the interval into finite subintervals such that each subinterval only contains one stationary point. The nth order Taylor polynomial of $e^{i\omega g(x)}$ is

$$F_0 = 1, \quad F_n(i\omega g(x)) = 1 + i\omega g(x) + \frac{(i\omega g(x))^2}{2!} + \frac{(i\omega g(x))^3}{3!} + \cdots + \frac{(i\omega g(x))^n}{n!}$$

and the nth order remainder of Taylor expansion is

$$T_n(x) = e^{i\omega g(x)} - F_n(i\omega g(x)). \tag{2.2}$$

$T_n(x)$ can be written as

$$T_n(x) = \begin{cases} \left(\cos(\omega g(x)) - 1 + \frac{(\omega g(x))^2}{2!} + \cdots + \frac{(-1)^{k+1}(\omega g(x))^{2k}}{(2k)!}\right) & + \\ i\left(\sin(\omega g(x)) - \omega g(x) + \cdots + \frac{(-1)^k(\omega g(x))^{2k-1}}{(2k-1)!}\right), & \text{n=2k}, \\ \left(\cos(\omega g(x)) - 1 + \frac{(\omega g(x))^2}{2!} + \cdots + \frac{(-1)^{k+1}(\omega g(x))^{2k}}{(2k)!}\right) & + \\ i\left(\sin(\omega g(x)) - \omega g(x) + \cdots + \frac{(-1)^k(\omega g(x))^{2k+1}}{(2k+1)!}\right), & \text{n=2k+1}. \end{cases}$$

Note that $U_n := \cos(x) - 1 + \frac{x^2}{2!} + \cdots + \frac{(-1)^{k+1}x^{2k}}{(2k)!}$, $V_n := \sin(x) - x + \cdots + \frac{(-1)^k x^{2k-1}}{(2k-1)!}$ are monotonic and smooth in $[0,+\infty)$ or $(-\infty,0]$ for all $n = 1,2,\ldots$. Hence, for monotonic and smooth function $g(x)$, $U_n(\omega g(x))$ and $V_n(\omega g(x))$ are smooth and monotonic in $[a,b]\cap[0,\infty)$ and $[a,b]\cap(-\infty,0]$. Therefore $T_n(x)$ are not oscillatory even if $e^{i\omega g(x)}$ is highly oscillatory for large ω. For example, $\cos(1000x^{\frac{1}{3}}) - 1$ is highly oscillatory, but $\cos(1000x^{\frac{1}{3}}) - 1 + \frac{(1000x^{\frac{1}{3}})^2}{2}$ and $\cos(1000x^{\frac{1}{3}}) - 1 + \frac{(1000x^{\frac{1}{3}})^2}{2} - \frac{(1000x^{\frac{1}{3}})^4}{4!}$ are monotonic and smooth.

For $g(x)$ having at most finite stationary points in $[a,b]$, by the intermediate value theorem for derivatives of Darbouxe [10], $g'(x)$ has the same sign between the stationary points and $g(x)$ is monotonic in these subintervals, the $nth(n \geq 2)$ Taylor remainders $T_n(x)$ is not oscillatory functions, either. To calculus the highly oscillatory integrals, we need only lower order Taylor expansions. Here we consider the first and second order Taylor expansions.